Department of Applied Physics
Entrance Examination Booklet

Physics I

(Answer the 2 Problems in this Booklet)

August 29 (Tuesday) 9:30 – 11:30, 2017

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the two problems in this booklet.
4. Use one answer sheet for each problem (two answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Let us consider the motions of a point mass with mass m and of a rigid-body sphere of unifrorm density with mass m and radius r placed on top of a semi-cylinder with radius R ($R > r$). The vertical and horizontal directions are defined as the directions of the z-axis and x-axis, respectively, and θ is defined as the angle between the z-axis and a straight line connecting the point mass or the center of the sphere with the origin as shown in Figures 1, 2 and 3. The center line of the semi-cylinder is fixed perpendicular to the xz-plane at the origin. The gravity acceleration is defined as g and points in the $-z$-direction.

[1] At some instant, the point mass starts to slide down from the top of the semi-cylinder and later lifts off at a critical angle θ_{c1} as shown in Figure 1. The friction between the semi-cylinder surface and the point mass is negligible. Moreover, the initial velocity of the point mass is zero. Answer the following questions.

[1.1] For $\theta < \theta_{c1}$, write the magnitude of the velocity v of the point mass as a function of θ.

[1.2] Determine $\cos\theta_{c1}$. Then, find the velocity v_{c1} of the point mass at $\theta = \theta_{c1}$.

[2] At some instant, the sphere starts to roll down from the top of the semi-cylinder and later lifts off from the semi-cylinder at a critical angle θ_{c2} as shown in Figure 2. Note that the sphere does not slip on the semi-cylinder surface and the rolling friction is negligible. Moreover, the initial velocity of the sphere is zero. Answer the following questions.

[2.1] Show that the moment of inertia of the sphere about an axis through its center is $\frac{2}{5}mr^2$.

[2.2] For $\theta < \theta_{c2}$, let us define the angular velocity of this sphere around its center and the velocity of the sphere center as ω and v, respectively. Show the relation between v and ω, remembering that the sphere does not slip on the semi-cylinder surface.

[2.3] Determine $\cos\theta_{c2}$. Then, find the velocity of the sphere center v_{c2} at $\theta = \theta_{c2}$.

[2.4] Explain the reason why θ_{c2} is different from θ_{c1} determined in Question [1.2].

[3] When the sphere is on top of the semi-cylinder, an impulse with magnitude P in the x-direction is applied to the sphere at $z = z_0$ as shown in Figure 3 ($R < z_0 < R + 2r$). After performing this experiment several times while varying z_0, it was found that the sphere starts to roll down the semi-cylinder without slipping for $z_0 = R+h$. After this, the sphere continues to roll down and lifts off at a critical angle θ_{c3}. Again, the rolling friction is negligible. Answer the following questions.

[3.1] Show the relation between h and r.

[3.2] Determine $\cos\theta_{c3}$.

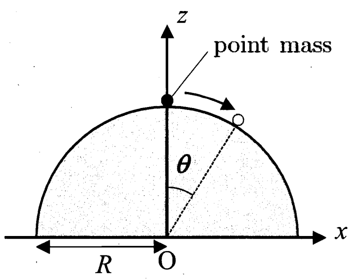

Figure 1

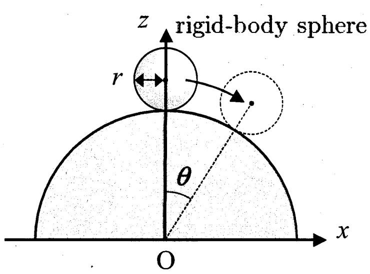

Figure 2

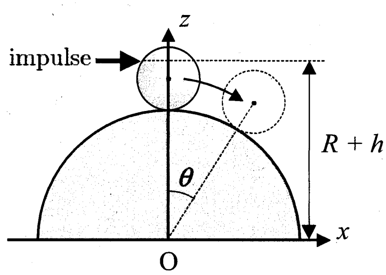

Figure 3

Problem 2

[1] Consider two concentric spherical shell conductors, Conductors A and B, of radii a and b, respectively, suspended in vacuum ($b > a$). As shown in Figure 1, they are connected to a power supply by lead wires coated with insulation film. The thickness of the conductors is negligibly small. Assume that the small opening in Conductor B introducing the lead wire and the lead wire itself do not affect their surroundings. Let ε_0 be the vacuum permittivity. Answer the following questions.

[1.1] The electric charges held by Conductors A and B are Q_0 and $-Q_0$, respectively ($Q_0 > 0$). Find the magnitude of the electric field $E(r)$ and the electric potential $\phi(r)$ at the distance r from the center of the spheres, with $\phi(\infty) = 0$.

[1.2] Find the capacitance between Conductor A and Conductor B.

[1.3] Consider the case that a medium of permittivity ε ($\varepsilon > 0$) uniformly fills the space between Conductors A and B. Find the electrostatic energy when Conductors A and B hold the electric charges Q_0 and $-Q_0$, respectively ($Q_0 > 0$).

[1.4] Consider the case that only a thin film of thickness d ($d \ll b - a$) and permittivity ε ($\varepsilon > 0$) is attached to the inner surface of Conductor B. Find the change in capacitance as compared to the result of Question [1.2] in the first order of d.

[2] Assume that the medium of permittivity ε ($\varepsilon > 0$) in Question [1.3] has a small electric conductivity σ. The frequency dependence of ε and σ can be ignored. Answer the following questions.

[2.1] A constant voltage is applied between Conductor A and Conductor B. After a sufficiently long time, they attain constant electric charges Q_0 and $-Q_0$, respectively ($Q_0 > 0$). For this condition, find the electric current flowing between Conductor A and Conductor B. Find also the electric resistance and the Joule heat generated per unit time. Let the positive direction of the electric current be the direction from Conductor A to Conductor B.

[2.2] In the circumstance of Question [2.1], Conductors A and B are separated from the power supply at $t = 0$. Find the time-dependent electric charge $Q(t)$ of Conductor A. Also find the Joule heat $W(t)$ that has been generated in the medium from $t = 0$ until a time t.

[2.3] Considering the results of Questions [1.3] and [2.2], explain the relation between the electrostatic energy and the Joule heat generated in the medium.

[3] Now, the medium in Question [2] between Conductors A and B is removed and replaced by two different kinds of media. Medium 1 of electric conductivity σ_1 and permittivity ε_1 ($\varepsilon_1 > 0$) fills the region $a < r < r_0$, whereas Medium 2 of electric conductivity σ_2 and permittivity ε_2 ($\varepsilon_2 > 0$) fills the region $r_0 < r < b$. Here, r is the distance from the center of the spheres. When a constant voltage is applied between the Conductors A and B, a constant electric current I flows after a sufficiently long time. For this condition, find the sheet density of electric charge that is accumulated at the boundary between Media 1 and 2.

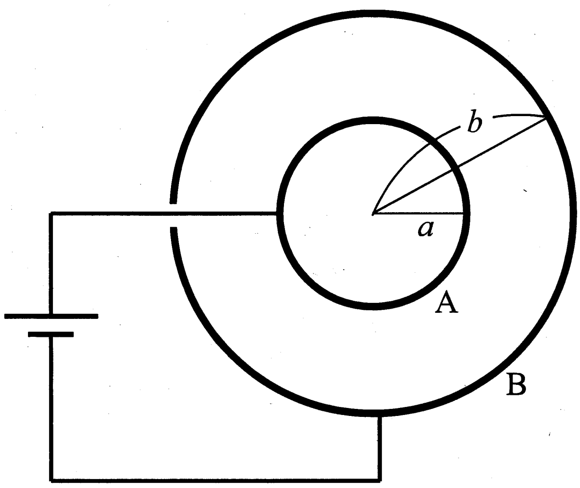

Figure 1

Department of Applied Physics
Entrance Examination Booklet

Physics II

(Answer 3 Problems among the 4 Problems in this Booklet)

August 29 (Tuesday) 13:00 – 16:00, 2017

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider the Schrödinger equation for particles with mass m moving along a straight line ($-\infty < x < \infty$). Below, $\hbar$ is the Planck constant divided by 2π, v is a positive constant, and $\delta(x)$ is the delta function.

[1] A solution with negative energy ($E < 0$) of the following one-dimensional one-particle Schrödinger equation,

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - v\delta(x)\right]\phi(x) = E\phi(x), \tag{1}$$

is given by $\phi(x) = Ne^{-\xi|x|}$, where N is a positive real normalization constant and ξ is a positive real number. Obtain N, ξ, and E.

[2] For a two-body system bound by a two-center potential, the one-dimensional two-particle Schrödinger equation is given as

$$\hat{H}\Phi(x_1, x_2) = E\Phi(x_1, x_2), \tag{2}$$

where the Hamiltonian $\hat{H}$ is defined as

$$\hat{H} = -\sum_{j=1,2}\frac{\hbar^2}{2m}\frac{d^2}{dx_j^2} - v\sum_{j=1,2}\delta(x_j + R/2) - v\sum_{j=1,2}\delta(x_j - R/2) + V(|x_1 - x_2|). \tag{3}$$

Here, $V(|x|)$ is a repulsive mutual interaction potential that assures the convergence of the integral $\int_{-\infty}^{+\infty} dxV(|x|)$. In the following, from Question [2.1] to [2.3], ignore whether the particles are bosons or fermions.

[2.1] Consider that the distance between the two potential centers R is much larger than the spread of the one-particle wave function $1/\xi$ and that the energy increase if both particles simultaneously approach one of the potential centers is much larger than the absolute energy value $|E|$ obtained in Question [1]. In this situation, the probability that the two particles are simultaneously observed at that potential center becomes very small. Then, we can take the following non-normalized trial wave function for the above two-particle Schrödinger equation (2),

$$\Phi_t(x_1, x_2) = c_1\phi_s(x_1 + R/2)\phi_s(x_2 - R/2) + c_2\phi_s(x_1 - R/2)\phi_s(x_2 + R/2), \tag{4}$$

where $\phi_s(x)$ is the solution of the the one-particle Schrödinger equation (1) in Question [1]. The coefficients c_1 and c_2 are real numbers.

Obtain the energy expectation value $E = E(c_1, c_2)$ of the trial wave function $\Phi_t(x_1, x_2)$ by using real numbers A, B, and S, defined as

$$A = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2\ \phi_s(x_1 + R/2)\phi_s(x_2 - R/2)\hat{H}\phi_s(x_1 + R/2)\phi_s(x_2 - R/2), \tag{5}$$

$$B = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2\ \phi_s(x_1 - R/2)\phi_s(x_2 + R/2)\hat{H}\phi_s(x_1 + R/2)\phi_s(x_2 - R/2), \tag{6}$$

$$S = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2\ \phi_s(x_1 - R/2)\phi_s(x_2 + R/2)\phi_s(x_1 + R/2)\phi_s(x_2 - R/2). \tag{7}$$

[2.2] Approximate solutions of the ground state of Equation (2) can be obtained through the variational principle applied to the energy expectation value of the trial wave function, $E = E(c_1, c_2)$, determined in Question [2.1]. Accordingly, extremes of $E = E(c_1, c_2)$ with respect to the variational parameters c_1 and c_2 give approximate values for the ground state energy, $\widetilde{E}$, and the corresponding wave functions $\widetilde{\Phi}(x_1, x_2)$. Showing your derivation, obtain the following set of linear equations for the variational parameters c_1 and c_2 that determine the extremes of $E = E(c_1, c_2)$. You may assume that $0 < S < 1$.

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} 1 & S \\ S & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{8}$$

[2.3] Obtain every solution $\widetilde{E}$ and the corresponding wave functions $\widetilde{\Phi}(x_1, x_2)$ of Equation (8) by using A, B, S, and $\phi_s(x)$. It is not necessary to normalize the wave functions.

[2.4] The wave functions obtained in Question [2.3] are called approximate solutions of the two-particle Schrödinger equation in Question [2.1]. Now, the two-particle Schrödinger equation considered in Question [2.1] to [2.3] shall describe fermions with spin angular momentum $\hbar/2$. The wave functions of the particles are functions of the position coordinates x_j and of the spin coordinates ω_j $(= \pm 1/2)$, where j $(= 1, 2)$ is the label of the particle. The spin wave functions $\alpha(\omega_j)$ and $\beta(\omega_j)$ represent the eigenfunctions of the z-component of the spin operator with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively. Derive every approximate solution of the two-particle Schrödinger equation by taking into account the spin degrees of freedom. The orbital part of the wave functions is supposed to be given by the solutions obtained in Question [2.3]. It is not necessary to normalize the wave functions.

Problem 2

Consider a lattice model for a system of volume V containing N monoatomic molecules in thermal equilibrium ($N \gg 1$). Divide this system into small cells with a constant volume of v, each of which can accommodate at most one molecule. The total number of cells in the system is $M = \frac{V}{v}$. For each cell i, define a variable σ_i so that $\sigma_i = 0$ when it is empty, and $\sigma_i = 1$ when it is occupied by a molecule. Each cell cannot contain two or more molecules (excluded volume effect), and each molecule can occupy only one cell. The microscopic state of the system is uniquely represented by a set of σ_i, $(\sigma_1, \sigma_2, ...\sigma_M)$. In the following, the density of molecules is expressed as $\phi = \frac{N}{M} = \frac{vN}{V}$, where $0 < \phi < 1$. Let T and k_B denote the temperature of the system and the Boltzmann constant, respectively.

[1] First, consider the case that molecules in different cells have no interaction.

[1.1] Calculate the entropy of the system $S = k_B \ln W$ (W is the number of states), using Stirling's formula for a large number N: $\ln N! \sim N \ln N - N$ (ln denotes the natural logarithm).

[1.2] Calculate the pressure of the above system, p_0, using the formula $p_0 = -\left.\frac{\partial F_0}{\partial V}\right|_{(T,N)}$, where F_0 is the Helmholtz free energy of the system. Compare p_0 to the ideal gas pressure, $p_{id} = \frac{k_B NT}{V}$, and explain the cause of any possible difference between p_0 and p_{id}. Remember that the cell volume v is constant. The following expansion may be used if necessary: $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$, $(0 \le x < 1)$.

[2] Next, consider the system with an attractive interaction α between the molecules in the nearest neighboring cells ($\alpha > 0$). The total energy of the system is expressed as

$$U = -\alpha \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{1}$$

where the summation $\sum_{\langle i,j \rangle}$ includes all nearest neighbor cell pairs. By replacing the individual σ_k with ϕ, the expected value of U in the thermal equilibrium state is approximated as

$$\overline{U}(\phi) = -\frac{1}{2} M z \alpha \phi^2 \qquad (z\text{: the number of nearest neighbor cells}). \tag{2}$$

[2.1] Let F be the Helmholtz free energy of the system. Calculate the chemical potential from $\mu = \frac{\partial F}{\partial N}$. Then derive the following expression:

$$\mu = -\alpha z \phi + k_B T \left[\ln \phi - \ln(1-\phi) \right]. \tag{3}$$

[2.2] Find the temperature range in which $\mu(\phi)$ derived in Question [2.1] is a monotonically increasing function of ϕ.

[2.3] If the temperature is out of the range determined in Question [2.2], there may exist three different ϕ's that give an identical value of μ. Explain the physical meaning of this situation, including the corresponding phenomena.

Problem 3

[1] Consider light-induced transitions in a non-degenerate two energy level system of atoms (Figure 1). Define Level 1 and Level 2 as the lower and upper energy levels, respectively, with the energy difference $\hbar\omega_0$. The Planck constant divided by 2π is denoted as $\hbar$. Define N_1 and N_2 as the atom numbers in Levels 1 and 2 per unit volume, respectively. The transition probability from Level 2 to Level 1 by spontaneous emission per unit time is denoted as A (Einstein coefficient A). When the transition from Level 2 to Level 1 occurs only by spontaneous emission, the time evolution of N_2 can be written as $\frac{dN_2}{dt} = -AN_2$. Under light with an angular frequency ω_0, transitions due to stimulated emission (Level 2 to Level 1 transition) and stimulated absorption (Level 1 to Level 2 transition) also occur. Probabilities per unit time for both stimulated emission and stimulated absorption are the same, and are defined as $BW(\omega_0)$ (B is called Einstein coefficient B). Here, $W(\omega_0)$ is the energy density of light per unit angular frequency and unit volume at ω_0. For simplicity, transitions due to stimulated emission and absorption occur only when the light angular frequency is at the resonance angular frequency ω_0. Answer the questions below.

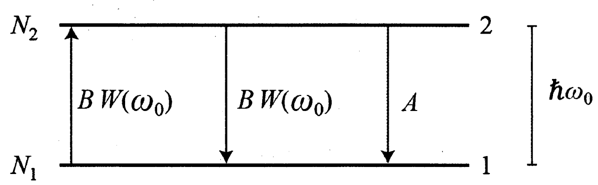

Figure 1

[1.1] Write simultaneous differential equations that show the time evolutions of N_1 and N_2.

[1.2] Express $W(\omega_0)$ in the steady state by A, B, N_1, and N_2. Show that $N_2/N_1 < 1$ is always satisfied.

[2] Under thermal equilibrium at a temperature T, the energy density of light per unit angular frequency and unit volume is described by the equation

$$W_{\text{eq}}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/(k_B T)} - 1}, \quad (1)$$

which is known as Planck's law of black-body radiation. Here, c is the speed of light in vacuum, and k_B is the Boltzmann constant. Let us derive $W_{\text{eq}}(\omega)$ by considering the number of modes of the electromagnetic waves, then obtain a relation between A and B. Answer the questions below.

[2.1] Find the number of modes of electromagnetic waves with angular frequencies from 0 to ω available in a cubic cavity of side length d.

[2.2] Find the available number of modes of electromagnetic waves per unit volume and unit angular frequency. Derive Equation (1) for Planck's black-body radiation. You may use that the average energy of an electromagnetic wave per mode is $\frac{\hbar\omega}{e^{\hbar\omega/(k_B T)}-1}$.

[2.3] Under the light energy density $W_{\rm eq}(\omega)$ that satisfies Planck's equation of black-body radiation (1), for any resonance angular frequency ω_0 in the two-level system of Question [1], N_1 and N_2 are distributed according to the Boltzmann distribution in thermal equilibrium. From $W(\omega_0) = W_{\rm eq}(\omega_0)$, derive a relation between the coefficients A and B as a function of ω_0.

[3] Consider now a non-equilibrium three-level system with optical pumping as shown in Figure 2. For simplicity, we consider only the transitions indicated by arrows in Figure 2. The energy difference between Level 1 and Level 3 is $\hbar\omega_p$, and light with an energy density of $W_p(\omega_p)$ per unit angular frequency and unit volume is used to pump from Level 1 to Level 3. The transition probabilities of both stimulated absorption and stimulated emission between Level 1 and Level 3 per unit time are $P = B_{13}W_p(\omega_p)$. The energy difference between Level 1 and Level 2 is $\hbar\omega_l$, and the transition probabilities of both stimulated absorption and stimulated emission between Level 1 and Level 2 per unit time are defined as $L = B_{12}W_l(\omega_l)$, under light whose energy density per unit angular frequency and unit volume is $W_l(\omega_l)$. The transition from Level 3 to Level 2 is non-radiative, and its transition probability per unit time is defined as C. The spontaneous emission probability per unit time from Level 2 to Level 1 and from Level 3 to Level 1 are defined as A and A', respectively. The atom number densities in Levels 1, 2, and 3 are defined as N_1, N_2, and N_3, respectively. A relation, $N_2/N_1 > 1$, between the atom number densities in Level 1 and Level 2 can be realized under a certain condition in the steady state. This is called population inversion, which is required for laser amplification of Level 2 to Level 1 transitions. Answer the questions bellow.

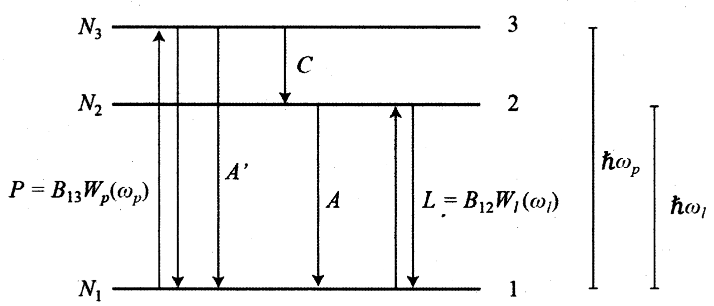

Figure 2

[3.1] Write simultaneous differential equations that show the time evolutions of N_1, N_2, and N_3.

[3.2] In the steady state, express N_2/N_1 by A, A', C, P, and L. When C and A fulfill a certain condition, there exists a threshold value, P_c, so that a population inversion is generated between Level 1 and Level 2 for $P > P_c$. Derive the necessary relation between C and A to generate a population inversion, and discuss its physical meaning. Determine P_c for this condition.

[3.3] Describe the reason why it becomes more difficult to realize a population inversion for shorter wavelength, assuming that the coefficient B does not depend on the transition energy.

Problem 4

Suppose that a charged particle with an effective mass m and an electric charge q moving in a conductor under a static electric field $\boldsymbol{E}$ and a static magnetic field (magnetic flux density $\boldsymbol{B}$) can be described by the following equation of motion,

$$m\left(\frac{d}{dt}+\frac{1}{\tau}\right)\boldsymbol{v}=q\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right), \tag{1}$$

where $\boldsymbol{v}$ is the velocity and τ is the relaxation time. Now, consider the steady state of a particle moving within a two-dimensional conductor with the velocity $\boldsymbol{v}=(v_x,v_y,0)^{\mathrm{T}}$ under $\boldsymbol{B}=(0,0,B)^{\mathrm{T}}$ $(B>0)$ and $\boldsymbol{E}=(E_x,E_y,0)^{\mathrm{T}}$. Here, T means transpose, i.e., the vectors are column vectors. In the following, consider only the x and y components of $\boldsymbol{E}$ and $\boldsymbol{v}$. Assume that m and τ are constants independent of energy and direction of motion, and ignore quantum mechanical effects.

[1] First, consider the situation that only electrons exist as charged particles. Let n be the sheet density and $q=-e$ the charge of an electron, where e is the elementary charge $(e>0)$. The cyclotron frequency is defined as $\omega_{\mathrm{c}}=eB/m$. Answer the following questions.

[1.1] Show that the current density $\boldsymbol{J}=-ne\boldsymbol{v}=(j_x,j_y)^{\mathrm{T}}$ is expressed as $\boldsymbol{J}=\tilde{\sigma}\boldsymbol{E}$ by using the electric field $\boldsymbol{E}=(E_x,E_y)^{\mathrm{T}}$ and the following conductivity tensor $\tilde{\sigma}$. Moreover, find σ_0.

$$\tilde{\sigma}=\frac{\sigma_0}{1+(\omega_{\mathrm{c}}\tau)^2}\begin{pmatrix}1 & -\omega_{\mathrm{c}}\tau\\ \omega_{\mathrm{c}}\tau & 1\end{pmatrix} \tag{2}$$

[1.2] When a current is applied along the x-direction of a two-dimensional conductor, which is electically open in the y-direction, as shown in Figure 1, $j_y=0$ is realized in the steady state. In this condition, find the Hall coefficient $R_{\mathrm{H}}=E_y/(j_xB)$ and the effective electrical conductivity along the x-direction $\sigma=j_x/E_x$.

[2] Next, consider the situation that both electrons and holes exist as charged particles. Let n and p be the sheet densities of electrons and holes and $-e$ and e the charges of an electron and a hole, respectively. Assume that the effective mass and the relaxation time of electrons and holes are the same, given as m and τ, respectively. Answer the following questions.

[2.1] Supposing that the total current is given by summation of the current carried by electrons and holes, find the conductivity tensor $\tilde{\sigma}$.

[2.2] Find the Hall coefficient R_{H} and the effective electrical conductivity along the x-direction σ in the steady state under the condition $j_y=0$.

[2.3] For R_{H} obtained in Question [2.2], let $R_{\mathrm{H}}^{(0)}$ and $R_{\mathrm{H}}^{(\infty)}$ be the Hall coefficient at the low magnetic field limit $(\omega_{\mathrm{c}}\tau\ll 1)$ and at the high magnetic field limit $(\omega_{\mathrm{c}}\tau\gg 1)$, respectively. Express p and n, respectively, in terms of $R_{\mathrm{H}}^{(0)}$ and $R_{\mathrm{H}}^{(\infty)}$. Note that $p\neq n$.

[2.4] For σ obtained in Question [2.2], sketch the $\omega_{\mathrm{c}}\tau$ dependence for $p/n=0$ and $p/n>0$ $(p\neq n)$, respectively, by considering the behavior at the low and high magnetic field limits.

[2.5] The situation $p=n$ can be realized in a certain type of semimetal. Describe how σ behaves at the high magnetic field limit and the physics behind this behavior.

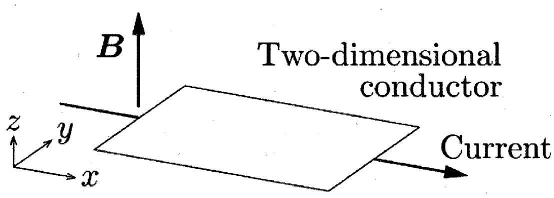

Figure 1