Department of Applied Physics
Entrance Examination Booklet

Physics

(Answer the 4 Problems in this Booklet)

August 29 (Tuesday) 9:00 – 13:00, 2023

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the four problems in this booklet.
4. Use one answer sheet for each problem (four answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the draft sheets of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

A magnetic material experiences a force in a non-uniform magnetic field. Based on this phenomenon, we construct a magnetometer to measure the magnetic moment of a sample. As shown in Fig. 1, we consider a cylindrical stick of uniform density, which never deforms, with radius a, length l_0, and mass M. We define two points on the central axis of the cylindrical stick: P at the end of the stick, and O away from P by the length l_1 ($> l_0/2$). We define x-, y-, and z-axes as shown in Fig. 1, where $+z$ is the vertical upward direction. Let the stick rotate with the axis of rotation being a line that passes through O and that is parallel to the y-axis. Let θ be the tilting angle of the stick measured from the vertical downward direction, as in Fig. 1. We neglect the air friction and the friction at O. Answer the following Questions.

[1] Firstly, we find the moment of inertia of the stick.

[1.1] Find the moment of inertia $I_{\parallel}$ when the axis of rotation is the central axis of the cylindrical stick.

[1.2] We create a sufficiently thin disk of thickness Δl by cutting the cylindrical stick perpendicular to its central axis. Find the moment of inertia $\Delta I_{\perp}$ for the disk when the axis of rotation passes through the center of mass and is perpendicular to the central axis of the cylindrical stick.

[1.3] In Fig. 1, find the moment of inertia I_{O} for the stick when the axis of rotation is a line parallel to the y-axis through O. If necessary, you can use $I_\beta = I_\alpha + M'R^2$ relating the moments of inertia I_α, I_β with respect to two axes α, β for an object of the mass M'. Here, α passes through the center of mass and β is parallel and shifted off by a distance R from α.

[2] A constant force $F_1(> 0)$ is applied along the x direction at P so that the stick keeps still at $\theta = \theta_0$ ($|\theta_0| < \pi/2$). Answer the following Questions. You can use I_{O} as the moment of inertia of the stick.

[2.1] Find $\tan\theta_0$. The gravitational acceleration constant is denoted by g.

[2.2] The stick starts to move when the force at P is abruptly removed. Find the absolute value of the angular velocity when the stick reaches $\theta = 0$. You can use θ_0.

[2.3] Find the period of this motion if θ_0 is sufficiently small.

[3] Next, we measure the magnetic moment of a magnetic sample using this stick. Let the stick be non-magnetic. As shown in Fig. 2, we set a sample at P. We neglect its size and mass. Using an electromagnet sufficiently far below O, we create a static magnetic field H along the x direction around the sample, with spatial gradient. We assume that the magnetic moment m of the sample points along the direction of H. In this case, the force F_1 along the x-axis is expressed as

$$F_1 = m\frac{\partial H}{\partial x}.$$

Here, assume $F_1 > 0$.

We keep $\theta = 0$ by adding the force $F_2(> 0)$ along the x-axis at the top of the stick. For this purpose, we mount a cylindrical iron core with radius b, magnetic permeability μ, and negligible mass at the top of the stick. Along the same axis, we also set a tightly-fitted solenoid of length X, the wire of which is wound uniformly with number of loops per length N. The iron core moves inside the solenoid without friction. Let a and b be sufficiently small as compared to l_0 and l_1. Answer the following Questions.

[3.1] In absence of the iron core and in presence of a steady current I through the wire, find the magnetic field inside the solenoid. Let X be sufficiently long, so that the magnetic field is uniform and parallel to the solenoid axis.

[3.2] The iron core is inserted into the solenoid up to a distance $X/3$. Find the inductance L of the solenoid. We neglect the magnetic flux and the inductance in the part of the solenoid where the iron core is not inserted, because μ is sufficiently much larger than the vacuum permeability μ_0.

[3.3] We measure F_1 through the force F_2 by which the iron core is pulled into the solenoid. Express F_1 in terms of μ, N, b, I, l_0, and l_1. You can use $F_2 = \frac{I^2}{2}\frac{\partial L}{\partial \xi}$ assuming that the magnetic field outside the solenoid is zero. Here, the iron core is inserted into the solenoid up to a distance ξ.

[3.4] The stick has $l_0 = 140$ cm and $l_1 = 90$ cm. The solenoid has the radius $b = 1.5$ cm and the number of loops per length $N = 2000\ \mathrm{m}^{-1}$. The iron core has the permeability $\mu = 6.4 \times 10^{-3}\ \mathrm{N{\cdot}A^{-2}}$. When the gradient of the magnetic field is 1600 Oe·cm^{-1}, we need the electric current $I = 200$ mA to keep $\theta = 0$. Therefore, the magnetic moment of the sample in this condition is [ ] Wb·m. Provide the number in [ ] with 2 significant digits. Beware that Oe is a CGS unit, that you need to convert 1 Oe $\rightarrow \frac{10^3}{4\pi}\ \mathrm{A{\cdot}m^{-1}}$ to obtain MKSA units, and that the unit of Wb is kg·m^2·A^{-1}·s^{-2}.

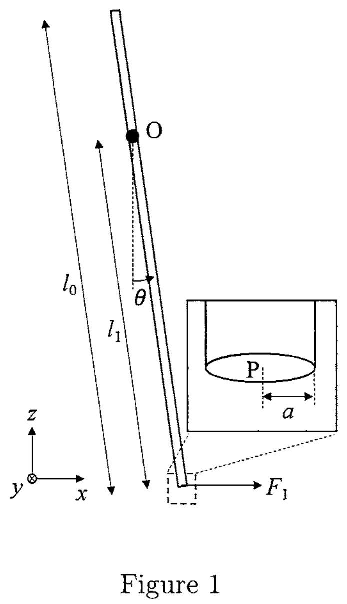

Figure 1

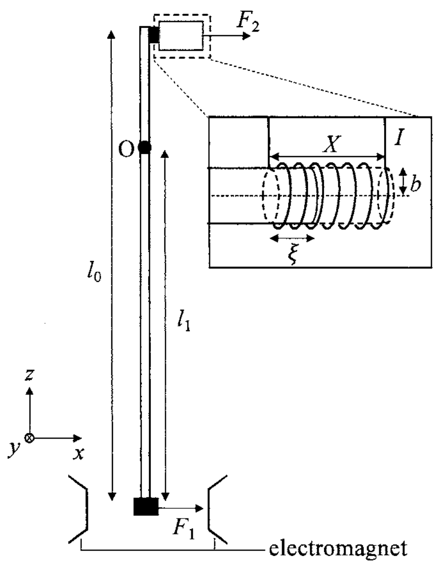

Figure 2

Problem 2

As shown in Fig. 1, two flat mirrors (Mirror 1 and Mirror 2) whose thickness is negligible are placed perpendicular to the z-axis in a vacuum. The amplitude reflection coefficients of opposing surfaces of Mirror 1 and Mirror 2 are r_1 and r_2, respectively. Between Mirror 1 and Mirror 2, a linearly polarized electromagnetic plane wave oscillating at a single angular frequency is propagating back and forth along the z-axis, forming a standing wave. We define the electric field amplitude at the antinode of the standing wave as E_0. Also, the distance between the mirrors is much longer than the wavelength of the electromagnetic wave. The area of the mirrors irradiated by the electromagnetic wave is large enough so that the plane wave approximation holds, and also sufficiently small as compared to the area of the mirrors.

Here, the dielectric constant of vacuum, the magnetic permeability of vacuum, and the speed of light in vacuum are ε_0, μ_0, and c, respectively.

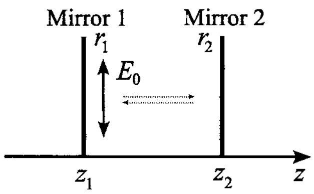

Figure 1

[1] Let us assume that both Mirror 1 and Mirror 2 are perfect conductors and that the amplitude reflection coefficients are -1 ($r_1 = r_2 = -1$). The positions of Mirror 1 and Mirror 2 are fixed at $z_1 = -\frac{L}{2}$ and $z_2 = \frac{L}{2}$ ($L > 0$), respectively. Answer the following Questions.

[1.1] In the space between Mirror 1 and Mirror 2, there exist only electromagnetic waves with discrete angular frequencies that can be specified by positive integers n. Derive an expression for the angular frequency ω_n.

[1.2] In complex number representation, the electric field and the magnetic flux density of the standing wave at time t and at the position z between Mirror 1 and Mirror 2 are given by $E_n(z,t) = \mathrm{Re}[\tilde{E}_n(z)e^{-i\omega_n t}]$ and $B_n(z,t) = \mathrm{Re}[\tilde{B}_n(z)e^{-i\omega_n t}]$, respectively. Express $\tilde{E}_n(z)$, $\tilde{B}_n(z)$ using E_0. Here, i is the imaginary unit.

[1.3] When the time average over one cycle is denoted as $\langle\cdots\rangle$, the time average of the momentum density of the electromagnetic wave $\langle \boldsymbol{g} \rangle$ is expressed as,

$$\langle \boldsymbol{g} \rangle = \frac{1}{c^2}\langle \boldsymbol{S} \rangle, \tag{1}$$

using the time average $\langle \boldsymbol{S} \rangle$ of the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. Express the pressure P exerted on Mirror 2 by the reflection of the electromagnetic wave between Mirror 1 and Mirror 2, using E_0. The electric field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$ can be written in terms of their complex counterparts, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$, as $\boldsymbol{E} = \mathrm{Re}[\tilde{\boldsymbol{E}}]$ and $\boldsymbol{H} = \mathrm{Re}[\tilde{\boldsymbol{H}}]$. Also, $\langle \boldsymbol{S} \rangle$ can be expressed as,

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathrm{Re}[\tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{H}}^*]. \tag{2}$$

Here, * denotes the complex conjugate.

[2] As shown in Fig. 2, we attach Mirror 2 to a wall-mounted spring whose spring constant is k. The length of the spring is the equilibrium length when the position of Mirror 2 is $z_2 = \frac{L}{2}$ ($L > 0$). Let us assume that both Mirror 1 and Mirror 2 are perfect conductors, that the amplitude reflection coefficients are -1 ($r_1 = r_2 = -1$), and that the position of Mirror 1 is fixed at $z_1 = -\frac{L}{2}$. Mirror 2 can move along the z-axis. When the electric field amplitude at the antinode of the standing wave is E_0 and the position of Mirror 2 is $z_2 = \frac{L}{2} + \delta$, Mirror 2 does not move and is in steady state. Assuming that the area of Mirror 2 irradiated by the electromagnetic wave is A, derive the sign of δ and express its magnitude using E_0.

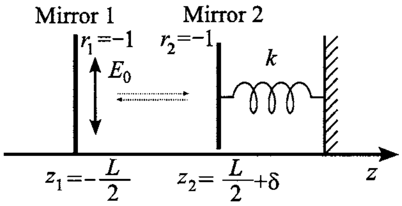

Figure 2

[3] As shown in Fig. 3, let us assume that the amplitude reflection coefficient of Mirror 1 is $r_1 = -r$ ($\frac{1}{2} < r < 1$) and its position is fixed as specified below. Mirror 2 is a perfect conductor and the amplitude reflection coefficient is $r_2 = -1$. Mirror 2 is attached to a spring as in the previous Question, and it can move along the z-axis. An electromagnetic plane wave propagating toward the $+z$ direction with angular frequency ω and electric field amplitude E_1 is incident on Mirror 1 from the back side of the mirror. In this case, the amplitude transmission coefficient of Mirror 1 is $\sqrt{1-r^2}$. Answer the following Questions.

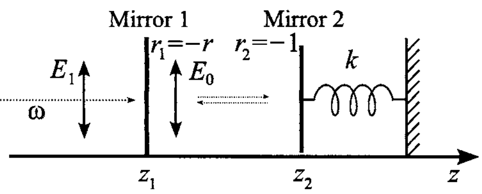

Figure 3

[3.1] We fix the positions of Mirror 1 at $z_1 = -\frac{L}{2}$ ($L > 0$) and of Mirror 2 at $z_2 = \frac{L}{2}$. Express the electric field amplitude at the antinode of the standing wave E_0 in the steady state, using E_1.

[3.2] We fix the position of Mirror 2 at $z_2 = \frac{L}{2}$. If we choose an appropriate ω and change the position z_1 of Mirror 1, $\frac{E_0}{E_1}$ takes the local maximum value at $z_1 = -\frac{L}{2}$. At $z_1 = -\frac{L}{2} \pm \Delta$ ($0 < \Delta < \frac{\pi c}{2\omega}$), $\frac{E_0}{E_1}$ becomes $\frac{1}{\sqrt{2}}$ of the local maximum value. Derive Δ. Here, we can use the inverse trigonometric functions.

[3.3] In the situation of the previous Question, we fix the position of Mirror 1 at either (a) $z_1 = -\frac{L}{2}$, (b) $z_1 = -\frac{L}{2} + \Delta$, or (c) $z_1 = -\frac{L}{2} - \Delta$. We allow Mirror 2 to move and to make a small-amplitude, free oscillation in the vicinity of the equilibrium point ($z_2 = \frac{L}{2} + \delta$). Answer which of (a), (b), (c) makes the effective spring constant larger than k. Also, describe qualitatively the reasons for this. Here, we assume that δ satisfies $|\delta| \ll \Delta$, and further that E_0 can instantaneously follow the change in the system.

Problem 3

Answer the following Questions on the specific heat (the heat capacity per unit mass) of solids due to lattice vibrations. Use $\hbar$ as the Planck constant divided by 2π, and $k_{\rm B}$ as the Boltzmann constant.

[1] First, we consider $M(\gg 1)$ independent classical harmonic oscillators in one dimension with mass m and natural angular frequency ω. The Hamiltonian of one harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{1}$$

with position x and momentum p.

[1.1] Obtain the partition function $Z(T) = \left(\frac{1}{2\pi\hbar}\int \mathrm{d}x\mathrm{d}p\, e^{-\beta H}\right)^M$ at temperature T. Here, $\beta = 1/(k_{\rm B}T)$ is the inverse temperature. You may use the formula $\int_{-\infty}^{\infty} e^{-t^2}\mathrm{d}t = \sqrt{\pi}$.

[1.2] Obtain the internal energy U.

[1.3] Obtain the heat capacity C.

[2] Next, we consider $M(\gg 1)$ independent quantum harmonic oscillators in one dimension with natural angular frequency ω. The eigenenergy of one harmonic oscillator is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{2}$$

with a non-negative integer n.

[2.1] Obtain the partition function $Z(T)$ at temperature T.

[2.2] Obtain the internal energy U.

[2.3] Obtain the heat capacity C.

[2.4] Express C's temperature dependence in a simple form at low temperatures, and obtain C's asymptotic value at high temperatures. Also, draw the schematic of the heat capacity C as a function of T.

[3] We consider the specific heat from lattice vibrations of atoms in three dimensional solids. We regard lattice vibrations as a collection of independent quantum harmonic oscillators (vibrational modes), and assume that the number of vibrational modes whose natural angular frequencies are between ω and $\omega + \mathrm{d}\omega$ is given by $g(\omega)\mathrm{d}\omega$ with

$$g(\omega) = \begin{cases} \dfrac{9N}{\omega_{\rm D}^3}\omega^2 & (\omega \le \omega_{\rm D}) \\ 0 & (\omega > \omega_{\rm D}). \end{cases} \tag{3}$$

Here, $\omega_{\rm D}$ is the Debye frequency and $N(\gg 1)$ is the number of atoms.

[3.1] Express the internal energy U and the heat capacity C as integrals with respect to ω.

[3.2] Obtain C's temperature dependence at low temperatures in the lowest order of T. In doing so, you may evaluate the integral by using $\int_0^\infty \frac{x^4 e^x}{(e^x-1)^2} dx = \frac{4\pi^4}{15}$. Also, obtain C's asymptotic value at high temperatures. In addition, draw the schematic of the heat capacity C as a function of T.

[4] The number of vibrational modes is almost $3N$ for three dimensional solids with $N(\gg 1)$ atoms. Let us compare (i) the heat capacity C_{I} of $3N$ one dimensional classical harmonic oscillators with a single natural angular frequency ω, (ii) the heat capacity C_{II} of $3N$ one dimensional quantum harmonic oscillators with a single natural angular frequency ω, and (iii) the heat capacity C_{III} obtained in [3].

[4.1] Plot $C_{\mathrm{I}}, C_{\mathrm{II}}$, and C_{III} as functions of T in a single graph.

[4.2] Compare and discuss $C_{\mathrm{I}}, C_{\mathrm{II}}$, and C_{III}'s behaviors at high temperatures. In particular, explain the physical reason for their similarity.

[4.3] Compare and discuss $C_{\mathrm{I}}, C_{\mathrm{II}}$, and C_{III}'s behaviors at low temperatures. In particular, explain the physical reason for their difference.

Problem 4

We consider a quantum optical system of a single optical mode with a second-order nonlinear optical effect. Let $\hbar$ be the Planck constant divided by 2π, $\omega(>0)$ be the angular frequency of the optical mode, $\hat{a}^\dagger$ and $\hat{a}$ be the photon creation and annihilation operators, respectively. The Hamiltonian of the system is given by

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\Delta}{2}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}), \tag{1}$$

where $\Delta(\geq 0)$ is a parameter measuring the strength of the nonlinear optical effect. Note that $\hat{a}^\dagger$ is the Hermitian conjugate of $\hat{a}$, and they satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{2}$$

Denoting the imaginary unit as i, we can define the following two observables from the creation and annihilation operators:

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{3}$$

[1] In absence of the nonlinear optical effect ($\Delta = 0$), the ground state of the Hamiltonian

$$\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a} \tag{4}$$

is the zero-photon vacuum state $|0\rangle$, which satisfies $\hat{a}|0\rangle = 0$.

[1.1] Evaluate the commutator $[\hat{x}, \hat{p}]$ using Eqs. (2) and (3).

[1.2] Using $\hat{a}|0\rangle = 0$, calculate the variances σ_x^2, σ_p^2 ($\sigma_O = \sqrt{\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2}$, $\langle ... \rangle = \langle 0|...|0\rangle$, $\hat{O} = \hat{x}, \hat{p}$) of the observables $\hat{x}, \hat{p}$ for the vacuum state $|0\rangle$. Also, evaluate $\sigma_x\sigma_p$, i.e., the product of the standard deviations, and then verify the uncertainty relation $\sigma_x\sigma_y \geq \frac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle|$.

[2] Next, we consider the general case with a nonlinear optical effect ($\Delta > 0$). To study the ground state of the system, it is helpful to exploit the following unitary transformation: Given a real parameter r, the unitary operator $\hat{S}(r) = e^{\frac{r}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)}$ transforms $\hat{a}, \hat{a}^\dagger$ as

$$\hat{S}(r)^\dagger\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{a}^\dagger\sinh r, \quad \hat{S}(r)^\dagger\hat{a}^\dagger\hat{S}(r) = \hat{a}^\dagger\cosh r - \hat{a}\sinh r. \tag{5}$$

Defining $\hat{H}(r) = \hat{S}(r)^\dagger\hat{H}\hat{S}(r)$ and given that Eq. (5) is a linear transformation of $\hat{a}^\dagger$ and $\hat{a}$, $\hat{H}(r)$ takes the following form:

$$\hat{H}(r) = \hbar\omega(r)\hat{a}^\dagger\hat{a} + \frac{\hbar\Delta(r)}{2}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}) + E(r). \tag{6}$$

[2.1] Using Eq. (5), find the expressions for $\omega(r), \Delta(r), E(r)$ in Eq. (6).

[2.2] With $\omega > \Delta$ assumed, the ground state of Eq. (1) turns out to be $\hat{S}(r)|0\rangle$ for some specific r. Express r in terms of ω, Δ. Also, explain qualitatively why the ground state has the form $\sum_{n=0}^{\infty} c_{2n}(\hat{a}^\dagger)^{2n}|0\rangle$ with coefficients c_{2n}, which consists of terms with even photon numbers.

[2.3] Assuming $\omega > \Delta$, express the variances σ_x^2, σ_p^2 of the observables $\hat{x}, \hat{p}$ for the ground state of Eq. (1) in terms of ω, Δ. Also, compare the results to those in Question [1.2] for the vacuum state, and verify the uncertainty relation.

[3] Finally, we consider the time evolution starting from the vacuum state $|0\rangle$ in the general case with a nonlinear optical effect. Suppose that the Hamiltonian is in a quadratic form of $\hat{a}, \hat{a}^\dagger$, as is the case of Eq. (1). The Heisenberg equation for $\hat{a}, \hat{a}^\dagger$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}(t) = \frac{i}{\hbar}[\hat{H}, \hat{O}(t)], \quad \hat{O}(0) = \hat{O}, \ \ \hat{O} = \hat{a}^\dagger, \hat{a} \tag{7}$$

turns out to be a linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \hat{a}(t) \\ \hat{a}^\dagger(t) \end{bmatrix} = -iM \begin{bmatrix} \hat{a}(t) \\ \hat{a}^\dagger(t) \end{bmatrix}, \tag{8}$$

where M is a 2×2 constant matrix.

[3.1] Find the matrix M corresponding to the Hamiltonian given in Eq. (1). If necessary, you may use $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$ for operators $\hat{A}, \hat{B}, \hat{C}$.

[3.2] Find the eigenvalues of M. Also, discuss the stability of the system in both of the parameter regimes $\omega < \Delta$ and $\omega > \Delta$.

[3.3] At the critical point $\omega = \Delta$, calculate the expectation value of the photon number $\hat{n} = \hat{a}^\dagger\hat{a}$ at time t. If necessary, you may use the fact that $e^A = I + A$ for a matrix A satisfying $A^2 = 0$. Here, I is the identity matrix.