Department of Applied Physics
Entrance Examination Booklet

Physics

(Answer the 4 Problems in this Booklet)

August 30 (Tuesday) 9:00 – 13:00, 2022

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the four problems in this booklet.
4. Use one answer sheet for each problem (four answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the draft sheets of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

We consider a quantum mechanical particle confined to an infinite-depth square-well potential in one or two dimensions. Let m represent the mass of the particle, i the imaginary unit, a a positive constant, and $\hbar$ the Planck constant divided by 2π. We neglect the spin degree of freedom, and wave functions are always normalized. You may use the following relations:

$$\begin{aligned}
\int_{-1}^{1} x\cos\left(\frac{\pi}{2}x\right)\sin(\pi x)\mathrm{d}x &= \frac{32}{9\pi^2}, \\
\int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right)\cos(\pi x)\mathrm{d}x &= \frac{4}{3\pi}, \\
\int_{-1}^{1} \sin\left(\frac{\pi}{2}x\right)\sin(\pi x)\mathrm{d}x &= \frac{8}{3\pi}.
\end{aligned}$$

[1] First, we consider the one-dimensional case. The time-independent Schrödinger equation is written as

$$\hat{H}\psi_n(x) = E_n\psi_n(x),$$

where E_n $(E_1 < E_2 < E_3\cdots)$ is the energy eigenvalue and $\psi_n(x)$ is the corresponding eigenfunction. It is noted that n is an integer greater than or equal to 1. The Hamiltonian $\hat{H}$ is expressed as

$$\begin{aligned}
\hat{H} &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \\
V(x) &= \begin{cases} 0 & (|x| \le a) \\ \infty & (|x| > a). \end{cases}
\end{aligned}$$

[1.1] Find $\psi_n(x)$ and E_n, where $\psi_n(x)$ is a real function.

[1.2] Let $\hat{P}$ be the operator which transforms the wave function $\psi(x)$ to $\psi(-x)$. Show that $\psi_n(x)$ is an eigenstate of $\hat{P}$ and find the corresponding eigenvalue.

[2] Next, we consider the two-dimensional case. The Hamiltonian $\hat{H}$ is expressed as

$$\begin{aligned}
\hat{H} &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y), \\
V(x,y) &= \begin{cases} 0 & (|x| \le a \text{ and } |y| \le a) \\ \infty & (|x| > a \text{ or } |y| > a). \end{cases}
\end{aligned}$$

For integers n_x and n_y greater than or equal to 1, a wave function written as $\Psi_{n_x,n_y}(x,y) = \psi_{n_x}(x)\psi_{n_y}(y)$ is an energy eigenstate of this system, where $\psi_n(x)$ are taken from Question [1]. The ground state is $\Psi_{1,1}(x,y)$, the first excited states are linear combinations of $\Psi_{2,1}(x,y)$ and $\Psi_{1,2}(x,y)$, and the second excited state is $\Psi_{2,2}(x,y)$.

[2.1] Let $\hat{M}$ be the operator which transforms the wave function $\Psi(x,y)$ to $\Psi(y,x)$. Show that the ground state and the second excited state are eigenstates of $\hat{M}$ and find the corresponding eigenvalues.

[2.2] Among the first excited states, $\Psi_{2,1}(x, y)$ and $\Psi_{1,2}(x, y)$ are not eigenstates of $\hat{M}$, but appropriate linear combinations of these states can be simultaneous eigenstates of $\hat{H}$ and $\hat{M}$. Find all such simultaneous eigenstates and identify the corresponding eigenvalue of $\hat{M}$ for each of them. Here, express the simultaneous eigenstates by linear combinations of $\Psi_{2,1}(x, y)$ and $\Psi_{1,2}(x, y)$, choosing the coefficient of $\Psi_{2,1}(x, y)$ to be a positive real number.

[2.3] For each of the states found in Question [2.2], calculate the expectation value of the angular momentum operator $\hat{L} = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$.

[2.4] Let $\hat{C}_4$ be the operator which transforms the wave function $\Psi(x, y)$ to $\Psi(-y, x)$. Show that the ground state and the second excited state are eigenstates of $\hat{C}_4$ and find the corresponding eigenvalues.

[2.5] Among the first excited states, $\Psi_{2,1}(x, y)$ and $\Psi_{1,2}(x, y)$ are not eigenstates of $\hat{C}_4$, but appropriate linear combinations of these states can be simultaneous eigenstates of $\hat{H}$ and $\hat{C}_4$. Find all such simultaneous eigenstates and identify the corresponding eigenvalue of $\hat{C}_4$ for each of them. Here, express the simultaneous eigenstates by linear combinations of $\Psi_{2,1}(x, y)$ and $\Psi_{1,2}(x, y)$, choosing the coefficient of $\Psi_{2,1}(x, y)$ to be a positive real number.

[2.6] For each of the states found in Question [2.5], calculate the expectation value of the angular momentum operator $\hat{L} = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$.

Problem 2

A parametric oscillation is an oscillatory phenomenon in the presence of a periodic modulation of a system parameter. We consider the system shown in Fig. 1 as a simple example for the parametric oscillation. A point mass m is suspended by a bar with length L and the supporting point oscillates along the vertical direction with an amplitude $a(\geq 0)$ and an angular frequency $\Omega(> 0)$. The position of the mass at time t is (x, y) in the cartesian coordinate frame. We define the origin of the coordinate frame as in Fig. 1. Note that the y-axis points downward in the vertical direction. The position of the supporting point is $(0, a\cos\Omega t)$. Let the mass of the bar be negligible, let its tension be $T(> 0)$, and let it never deform. The angle θ is measured from the vertical direction as in Fig. 1. The gravitational acceleration constant is denoted by $g(> 0)$. We neglect the friction at the supporting point and the air friction. In the following, the dot superscript indicates the time derivative.

[1] Express the position (x, y) of the mass at time t using L, θ, a, Ω, and t.

[2] Express the equations of motion for (x, y) using $\ddot{x}, \ddot{y}, m, T, \theta$, and g.

[3] By differentiating the results of Question [1], express $\ddot{x}$ and $\ddot{y}$ using $L, \theta, \dot{\theta}, \ddot{\theta}, a, \Omega$, and t.

[4] Write down the equation of motion for θ. The tension T should not appear in the answer.

[5] Assuming that θ is small, we approximate $\sin\theta$ as θ. We use this approximation in the following questions. Determine the solution $\theta_0(t)$ of the equation of motion derived in Question [4] when $a = 0$, $\theta_0(0) = A$, and $\dot{\theta}_0(0) = 0$. Also, determine the characteristic angular frequency $\omega_0(> 0)$ using g and L.

[6] We set the solution for small a to be $\theta(t) = \theta_0(t) + a\theta_1(t)$ using $\theta_0(t)$ derived in Question [5]. Assuming that the solution can be determined by perturbative expansion in a, find the equation for $\theta_1(t)$ in the first-order approximation.

[7] We want to obtain the solution $\theta_1(t)$ of the equation derived in Question [6]. We assume the following form of a special solution with time-independent constants u_1 and u_2 and the characteristic angular frequency ω_0, which was obtained in Question [5]:

$$\theta_1(t) = u_1 \cos[(\Omega + \omega_0)t] + u_2 \cos[(\Omega - \omega_0)t].$$

Express the condition for the existence of this special solution using Ω, g, and L. In addition, determine u_1 and u_2 under this condition, and express them using only L, A, Ω, and g.

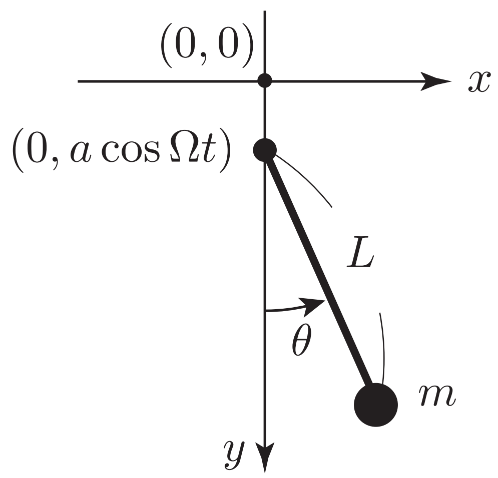

Figure 1

Problem 3

A container of a large heat capacity and a large mass has an internal cylindrical cavity, which holds a gas of $N(\gg 1)$ identical particles. Let L be the height of the cavity, R be its radius, and define $\nu = N/L$. We use a cartesian coordinate system xyz at rest with its origin O located at the center of the cavity and its z axis aligned with the axis of rotational symmetry (Figure 1). The Boltzmann constant and the Planck constant are denoted by k and h, respectively.

In the following, we refer to the z component of an angular momentum vector simply as angular momentum. We refer to the pressure exerted on the internal side of the container as side pressure. The natural logarithm of a positive number t is denoted by $\log t$.

We represent the position of a particle by vector $\boldsymbol{r} = (x, y, z)$ and the corresponding momentum by vector $\boldsymbol{p} = (p_x, p_y, p_z)$. The Hamiltonian of one particle is given by $H_1(\boldsymbol{r}, \boldsymbol{p}) = (p_x^2 + p_y^2 + p_z^2)/(2m)$, where m is its mass, and its angular momentum is given by $M_1(\boldsymbol{r}, \boldsymbol{p}) = xp_y - yp_x$. The Hamiltonian of N particles is given by the sum of one-particle Hamiltonians. Let $\rho(\boldsymbol{r}, \boldsymbol{p})$ denote the probability density function for the distribution of one particle in phase space.

Treat the motion of the particles in classical mechanics and neglect the effect of gravity on the particles. You may use the improper integral $\int_{-\infty}^{\infty} e^{-t^2} \mathrm{d}t = \sqrt{\pi}$.

[1] Consider the case where the container is at rest, fixed to the floor, and is regarded as a heat bath of temperature T (Figure 2). The container and the gas are in thermal equilibrium. The radius R is sufficiently large and the motion of the N particles follows a canonical distribution. The probability density function $\rho(\boldsymbol{r}, \boldsymbol{p})$ of one particle is then given by

$$\rho(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{h^3 L \tilde{Z}_1} \exp\left(-\frac{H_1(\boldsymbol{r}, \boldsymbol{p})}{kT}\right),$$

where $\tilde{Z}_1$ is a quantity that depends neither on $\boldsymbol{r}$ nor on $\boldsymbol{p}$. Answer the following questions.

[1.1] The quantity $\tilde{Z}_1$ can be regarded as a function of temperature T and radius R. Using the normalization condition of the probability density function $\rho(\boldsymbol{r}, \boldsymbol{p})$,

$$\int_{-\infty}^{\infty} \mathrm{d}p_x \int_{-\infty}^{\infty} \mathrm{d}p_y \int_{-\infty}^{\infty} \mathrm{d}p_z \int_{-L/2}^{L/2} \mathrm{d}z \int_{x^2+y^2 \le R^2} \mathrm{d}x\mathrm{d}y \; \rho(\boldsymbol{r}, \boldsymbol{p}) = 1,$$

calculate $\tilde{Z}_1(T, R)$ and express it in terms of kT, R, and $\gamma = \pi^{5/2}(2m)^{3/2}h^{-3}$.

[1.2] Let F be the Helmholtz free energy of this gas and define $\tilde{F} = F/L$. By considering the partition function of N particles in the limit of $N, L \to \infty$ at fixed $\nu = N/L$, derive $\tilde{F}$ and express it in terms of kT, ν, and $\tilde{Z}_1(T, R)$. You may use the approximation $\log N! \approx N \log N - N$.

[1.3] By considering the work required for a virtual infinitesimal change in radius R of the cavity, calculate the side pressure P from the free energy F in Question [1.2] and express it in terms of $kT, R, \nu, \tilde{Z}_1(T, R)$, and the partial derivative $\left(\frac{\partial \tilde{Z}_1(T,R)}{\partial R}\right)_T$. In addition, express P in terms of kT, R, and ν by replacing $\tilde{Z}_1(T, R)$ with the answer to Question [1.1].

[2] Consider the case where the container is suspended at the center of its top surface and is allowed to rotate freely around the z axis without friction (Figure 3). Assume that the container has rotational symmetry around the z axis. Suppose that the container is rotating around the z axis at a constant angular velocity ω and that its temperature is T. The container and the gas are in thermal equilibrium and the probability density function $\rho(\boldsymbol{r},\boldsymbol{p})$ of one particle is given by

$$\rho(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{h^3 L \tilde{Z}_1} \exp\left(-\frac{H_1(\boldsymbol{r},\boldsymbol{p}) - \omega M_1(\boldsymbol{r},\boldsymbol{p})}{kT}\right),$$

where $\tilde{Z}_1$ is a quantity that depends neither on $\boldsymbol{r}$ nor on $\boldsymbol{p}$. Answer the following questions.

[2.1] Noting that the position $\boldsymbol{r}$ is confined to the cavity, derive the minimum value of $H_1(\boldsymbol{r},\boldsymbol{p}) - \omega M_1(\boldsymbol{r},\boldsymbol{p})$ in phase space and express it in terms of R, ω, and m.

[2.2] Let $(v_x(\boldsymbol{r}), v_y(\boldsymbol{r}), v_z(\boldsymbol{r}))$ be the expectation value of the velocity vector of a particle in the vicinity of position $\boldsymbol{r}$. Derive $v_x(\boldsymbol{r})$ and $v_y(\boldsymbol{r})$.

[2.3] The quantity $\tilde{Z}_1$ can be regarded as a function of temperature T, radius R, and angular velocity ω. Calculate $\tilde{Z}_1(T,R,\omega)$ in this case and express it in terms of kT, R, m, ω, and $\gamma = \pi^{5/2}(2m)^{3/2}h^{-3}$.

[2.4] Express the side pressure P in terms of kT, R, ω, ν, and m. In addition, derive the value of P in the limit of low temperature at fixed angular velocity ω.

[2.5] Calculate the moment of inertia I of the gas around the z axis and express it in terms of kT, ω, N, and $\varepsilon = mR^2\omega^2/2$.

[2.6] Calculate the heat capacity C_ω of the gas at constant angular velocity ω and express it in terms of k, T, N, and $\varepsilon = mR^2\omega^2/2$. In addition, derive the values of heat capacity C_ω in the limits of high temperature and of low temperature.

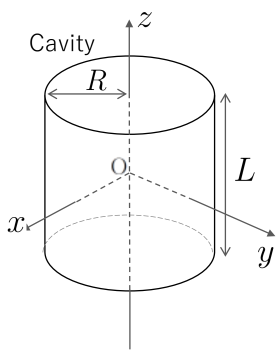

Figure 1

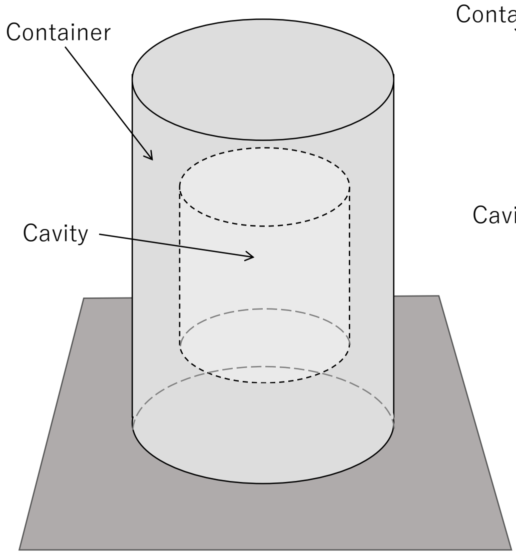

Figure 2

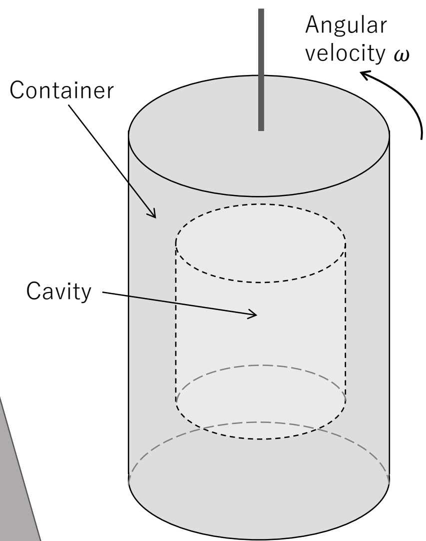

Figure 3

Problem 4

We consider a classical charged particle with mass m and charge q (>0) in vacuum, whose position coordinate at time t is given by $\boldsymbol{r}(t) = (x(t), y(t), z(t))$. We study ways to confine it in a bounded region around the origin $(0,0,0)$ by using electromagnetic fields. Suppose that the velocity of the charged particle is sufficiently small compared to the speed of light, and the effect of the electromagnetic field generated by the charged particle itself can be neglected. Use i as the imaginary unit and answer the following questions.

[1] First, suppose that a static electric field is imposed so that the scalar potential at position $\boldsymbol{r} = (x, y, z)$ is given by

$$\phi(\boldsymbol{r}) = ax^2 + by^2 + cz^2. \tag{1}$$

Here, a, b, and c are constants. The electric field is given by $\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r})$.

[1.1] The charged particle can be confined in a bounded region if a, b, and c of the scalar potential in Eq. (1) are all positive. However, it is impossible to impose a static electric field that makes all of a, b, and c positive since $a+b+c=0$ holds in reality. Show $a+b+c=0$ in Eq. (1).

Below, let us consider a way to confine the charged particle in a bounded region by imposing a uniform static magnetic field in addition to the scalar potential in Eq. (1). Suppose that $a = b = -\frac{c}{2} < 0$ in Eq. (1), and consider the situation where a static magnetic field with magnetic flux density $\boldsymbol{B} = (0, 0, B)$ is imposed.

[1.2] Write down the equation of motion for the position $\boldsymbol{r}(t)$ of the charged particle in terms of $\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}$, $\frac{\mathrm{d}^2\boldsymbol{r}(t)}{\mathrm{d}t^2}$, m, q, $\phi(\boldsymbol{r}(t))$, and $\boldsymbol{B}$ by considering the force acting on the charged particle from the static electric field and the static magnetic field.

[1.3] The charged particle is confined along the z axis and undergoes harmonic oscillations. Express its angular frequency in terms of m, q, and c.

[1.4] By defining $u(t) = x(t) + iy(t)$, express the equation of motion for $u(t)$ in terms of $u(t)$, $\frac{\mathrm{d}u(t)}{\mathrm{d}t}$, $\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}$, m, q, c, and B.

[1.5] Find the condition for m, q, c, and B under which the charged particle is confined in a bounded region along the x and y axes for any initial condition.

[2] Next, suppose that, instead of Eq. (1), the scalar potential at position $\boldsymbol{r} = (x, y, z)$ is given by

$$\tilde{\phi}(\boldsymbol{r}, t) = (\alpha x^2 + \beta y^2 + \gamma z^2)\cos\omega t \tag{2}$$

and varies with time at angular frequency ω. Here, α, β, and γ are constants. In this case, under an appropriate condition, the charged particle can be confined in a bounded region along all three axes x, y, and z. Below, let us explain this phenomenon by focusing on the x component of the motion of the charged particle. Note that the magnetic field can be neglected and the electric field is given by $\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = -\nabla\tilde{\phi}(\boldsymbol{r}, t)$.

[2.1] The x component of the position of the charged particle can be expressed as a function of τ by $x(\tau)$, where $\tau = \dfrac{\omega t}{2}$. In this case, the equation of motion for $x(\tau)$ can be written as

$$\frac{\mathrm{d}^2 x(\tau)}{\mathrm{d}\tau^2} + 2\lambda x(\tau)\cos 2\tau = 0, \tag{3}$$

where λ is a constant. Find λ.

[2.2] Suppose that the solution of Eq. (3) is given by

$$x(\tau) = \sum_{n=-\infty}^{\infty} C_n \cos\left[(2n+\Omega)\tau\right] \tag{4}$$

for an appropriate initial condition. Here, Ω and C_n (n is an integer) are independent of τ, and $0 < \Omega < 1$ holds. Express the relation between Ω, C_{n-1}, C_n, and C_{n+1} in terms of λ and n.

[2.3] When $0 < \lambda \ll 1$ in Question [2.2], the contribution of C_n in Eq. (4) can be neglected except for C_{-1}, C_0, and C_1. As an approximation, we use the relation in Question [2.2] for $n = -1, 0, 1$ only. Under this condition and under the assumption $C_0 \neq 0$, we obtain $\dfrac{C_{-1}}{C_0} = \dfrac{C_1}{C_0} = \varepsilon\lambda$ and $\Omega = \delta\lambda$ up to first order in λ, where ε and δ are constants. Find ε and δ.

[2.4] Express the solution $x(\tau)$ of Eq. (3) in terms of C_0, λ, and τ, using the approximations in Question [2.3]. When $C_0 > 0$, draw a sketch of $x(\tau)$ as a function of τ.