Department of Applied Physics
Entrance Examination Booklet

Physics

(Answer the 4 Problems in this Booklet)

August 31 (Tuesday) 9:00 – 13:00, 2021

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the four problems in this booklet.
4. Use one answer sheet for each problem (four answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the draft sheets of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider a spin 1/2 particle with magnetic moment $\mu(<0)$ placed in a static magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ along the z direction. By applying a weak rotating magnetic field $\boldsymbol{B}_1(t) = B_1(\boldsymbol{e}_x \cos\omega t + \boldsymbol{e}_y \sin\omega t)$ with angular frequency $\omega(>0)$, let us determine the transition frequency $\omega_0 = -2\mu B_0/\hbar$ between the spin-eigenstates $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which represent projections on the z axis. Here, $\boldsymbol{e}_x, \boldsymbol{e}_y$, and $\boldsymbol{e}_z$ are the unit vectors along the x, y, and z directions, respectively, $h = 2\pi\hbar$ is the Planck constant, while $B_0 > 0$ and $B_1 > 0$. The Hamiltonian for a spin 1/2 particle with magnetic moment μ placed in the magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$ is given by

$$\hat{H} = -\mu \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}.$$

Answer the following questions. Use i as the imaginary unit and $\omega_1 = -2\mu B_1/\hbar$.

[1] The time-dependent Schrödinger equation is given by

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$

The spin state of the particle at time t is given by $|\psi(t)\rangle = \alpha(t)|+\rangle + \beta(t)|-\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$.

[1.1] Express the Hamiltonian for the particle in the magnetic field $\boldsymbol{B}(t) = \boldsymbol{B}_0 + \boldsymbol{B}_1(t)$ in terms of ω, ω_0, ω_1, and t. Find differential equations for $\alpha(t)$ and $\beta(t)$.

[1.2] Let us introduce new variables $a(t) = \alpha(t)e^{i\omega_0 t/2}$ and $b(t) = \beta(t)e^{-i\omega_0 t/2}$. They satisfy the following differential equations,

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\omega_1}{2}\begin{pmatrix} 0 & A(t) \\ A^*(t) & 0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

Find $A(t)$.

[2] Suppose the spin state of the particle is $|-\rangle$ at $t = 0$. Between $t = 0$ and $t = \tau(\gg \omega_0^{-1})$, we apply the weak, rotating magnetic field $\boldsymbol{B}_1(t)$ in addition to the static magnetic field $\boldsymbol{B}_0$.

[2.1] Calculate the probability $P_1 = |a(\tau)|^2$ to find the particle in the $|+\rangle$ state at $t = \tau$. Express P_1 in terms of ω, ω_0, ω_1 and τ. Note that the transition probability induced by the rotating magnetic field $\boldsymbol{B}_1(t)$ is so small that you can approximate $b(t) = 1$. Use $1 - \cos\theta = 2\sin^2(\theta/2)$, if necessary.

[2.2] Draw a sketch of P_1 as a function of ω near $\omega = \omega_0$. In the sketch, indicate the maximum value of P_1. Also indicate the angular frequencies ω that satisfy $P_1(\omega) = 0$ and are the nearest neighbors to ω_0. The probability of the excited state as a function of ω is referred to as an excitation spectrum.

[3] After the manipulations in [2], we apply the rotating magnetic field $\boldsymbol{B}_1(t)$ from $t = T(\gg \tau)$ to $t = T + \tau$. Namely, the magnetic field applied to the particle is given by (Fig. 1),

$$\boldsymbol{B}(t) = \begin{cases} \boldsymbol{B}_0 + \boldsymbol{B}_1(t) & 0 \le t < \tau \\ \boldsymbol{B}_0 & \tau \le t < T \\ \boldsymbol{B}_0 + \boldsymbol{B}_1(t) & T \le t < T + \tau. \end{cases}$$

[3.1] Calculate the probability $P_2 = |a(T+\tau)|^2$ to find the particle in the $|+\rangle$ state at $t = T+\tau$. Express P_2 in terms of ω, ω_0, ω_1, τ, and T. Note that the transition probability induced by the rotating magnetic field $\boldsymbol{B}_1(t)$ is so small that you can approximate $b(t) = 1$. Use $1 + \cos\theta = 2\cos^2(\theta/2)$, if necessary.

[3.2] Draw a sketch of P_2 as a function of ω near $\omega = \omega_0$. Find the maximum value of P_2. Also find the angular frequencies $\omega = \Omega_1$, Ω_2 that satisfy $P_2(\omega) = 0$, where Ω_1 and Ω_2 are the nearest neighbors to ω_0 and $\Omega_1 < \omega_0 < \Omega_2$ holds.

[3.3] Let us define the precision of a measurement of ω_0 by the spectral width $\Delta\Omega = \Omega_2 - \Omega_1$. Find $\Delta\Omega$ and discuss how to improve the measurement precision of ω_0.

[4] Consider the excitation spectrum in [3] by analogy with an interference fringe obtained in Young's double slit experiment using slits of width D separated by a distance l (see Fig. 2). Young's experiment shows the interference pattern after measuring many photons. Similarly, one can obtain the excitation spectrum given in [3.2] by varying ω and repeating the measurement of [3].

[4.1] Identify parameters in the measurement [3] that correspond to D and l in Young's experiment. For a light wave with wavelength λ and a screen placed at a distance $L(\gg l,\, D)$ from the slit, the separation between the first dark fringes is $\lambda L/l$ for Young's experiment, and that for a single slit of width D is $2\lambda L/D$.

[4.2] Suppose that the z component of the spin is measured at time $t\,(\tau < t < T)$ in [3]. Illustrate the excitation spectrum observed at $t = T + \tau$ after repeating this experiment many times.

Figure 1

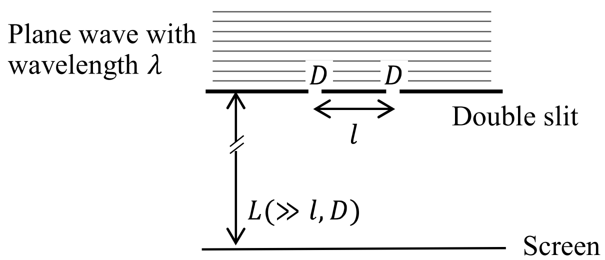

Figure 2

Problem 2

Let us consider a one-dimensional system in which mass points and springs are aligned in alternating fashion. The natural length of each spring is a and the spring constant is k. Initially, the mass points are placed at every position of the form $x = na$ (n is an integer) and we index the mass point at $x = na$ as the n-th mass point. The displacement of the n-th mass point from $x = na$ at time t is denoted by $x_n(t)$. In the following, i is the imaginary unit. Answer the questions below.

[1] First, suppose that the mass of each mass point is m (Fig. 1). The numbers of mass points and springs are sufficiently large so that edge effects can be neglected.

[1.1] Write down the equation of motion for the n-th mass point.

[1.2] Assume the form $x_n(t) = e^{iqn}c_q(t)$ $(0 \le q < 2\pi)$. Substitute this expression into the equation of motion in [1.1], and derive the differential equation for $c_q(t)$. Note that the actual displacement of the n-th mass point is given by the real part of $x_n(t)$.

[1.3] When $0 < q < 2\pi$, the general solution to the differential equation in [1.2] is given as a linear combination of $e^{+i\omega_q t}$ and $e^{-i\omega_q t}$. Express ω_q in terms of m, k, and q, provided $\omega_q > 0$.

[1.4] When $q = 0$, find the general solution to the differential equation in [1.2].

[2] Now, suppose that the mass of the 0-th mass point in the system of [1] is modified to M (Fig. 2). The masses of all other mass points and the spring constants are unchanged. Consider a situation where the oscillation of the mass points $x_n(t) = e^{iqn-i\omega_q t}$ $(0 < q < \pi)$ is incident from $x < 0$. To find the transmission amplitude T_q and the reflection amplitude R_q, we assume the form

$$x_n(t) = \begin{cases} e^{iqn-i\omega_q t} + R_q e^{-iqn-i\omega_q t} & (n \le -1) \\ T_q e^{iqn-i\omega_q t} & (n \ge 0) \end{cases} \tag{1}$$

as a solution to the equations of motion. Note that, in general, T_q and R_q are complex numbers.

[2.1] Write down the equation of motion for each mass point. Pay attention to the treatment of the 0-th mass point.

[2.2] Substitute the assumed form (1) into the equations of motion for $n \le -2$ or $n \ge +1$, and show that the expression for ω_q is unchanged from the one derived in [1.3] even when $M \ne m$.

[2.3] Based on the result of [2.2], substitute the assumed form (1) into the equations of motion for $n = 0$ and $n = -1$, and derive simultaneous equations for T_q and R_q.

[2.4] Using the simultaneous equations in [2.3], show that $R_q = T_q - 1$. Furthermore, express the transmission amplitude T_q in terms of M, m, and q.

[2.5] Set $M = m$ and $M = +\infty$ in the solution in [2.4] and interpret the results.

[3] Next, let us impose periodic boundary conditions on the system considered in [2] (Fig. 3). The total numbers of mass points and springs are both L, which is assumed to be an even integer. The mass point at $n = L/2$ is connected to the mass point at $n = -(L/2) + 1$ by a

spring. To find the normal modes of this system, we consider incident oscillations from $x < 0$ and $x > 0$ simultaneously. We assume the form

$$x_n(t) = \begin{cases} A_q e^{iqn - i\omega_q t} + B_q e^{-iqn - i\omega_q t} & (n \le -1) \\ C_q e^{iqn - i\omega_q t} + D_q e^{-iqn - i\omega_q t} & (n \ge 0) \end{cases} \tag{2}$$

with $0 < q < \pi$. Because the equations of motion are linear and have inversion symmetry about $x = 0$, the results of [2] can be expressed as

$$\begin{pmatrix} C_q \\ B_q \end{pmatrix} = S_q \begin{pmatrix} A_q \\ D_q \end{pmatrix}, \quad S_q = \begin{pmatrix} T_q & R_q \\ R_q & T_q \end{pmatrix}. \tag{3}$$

[3.1] The periodic boundary condition implies $A_q = C_q e^{iqL}$ and $B_q = D_q e^{-iqL}$. Requiring that these conditions and Eq. (3) are satisfied simultaneously, derive the equation that determines the allowed values of q in this system.

[3.2] When $M = m$, express the allowed values of q in terms of L. Also, find the corresponding normal frequency ω_q.

[3.3] When $M = 2m$, express the allowed values of q in terms of L. Also, find the corresponding normal frequency ω_q. Recalling the condition $0 < q < \pi$, count the number of normal modes which can be given in the form of Eq. (2). If their number is less than L, describe the remaining mode(s).

[3.4] Discuss the case $M \to \infty$ in the same way as in [3.3].

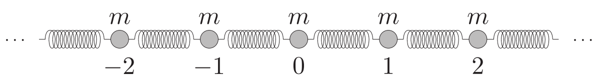

Figure 1

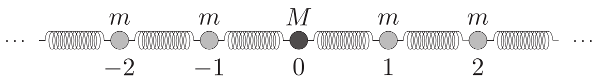

Figure 2

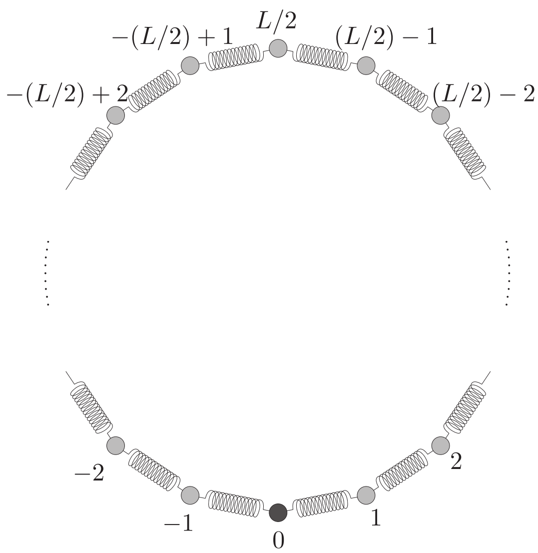

Figure 3

Problem 3

Let us consider a binary alloy that consists of two types of atoms, A and B. Let the lattice have N sites, where each site is occupied by either an A atom or a B atom (see Fig. a). The number of A atoms is N_A, the number of B atoms is N_B, and $N = N_A + N_B$. We assume that $N_A \gg 1$ and $N_B \gg 1$. When the number of nearest neighbor pairs that are occupied by different kinds of atoms is N_{AB}, the internal energy E is given by

$$E = N_{AB} V, \tag{1}$$

with $V > 0$. The number of nearest neighbors (the coordination number) z is the same for each site.

First, let us consider a uniform alloy (as shown in Fig. b). We define the parameter x that determines the composition ratio of the alloy by writing $N_A = xN$, and $N_B = (1-x)N$ with $0 < x < 1$. The Boltzmann constant is k_B. Answer the following questions.

[1] A site occupied by an A atom has z nearest neighbor sites. We approximate that B atoms occupy those nearest neighbor sites with the probability N_B/N. With this approximation, express N_{AB} in terms of z, N, and x. In addition, express the internal energy E in terms of z, N, x, and V.

[2] Suppose that the entropy S of the alloy is determined exclusively by the configuration of the atoms. Express S in terms of k_B, N, and x. In doing so, use Stirling's formula, $\log N! \simeq N \log N - N$ for $N \gg 1$.

[3] Find the Helmholtz free energy F at a temperature T.

Next, let us consider alloys with various composition ratios. To this end, we treat x as a variable. We define the average free energy per site, $f(x) = F/N$, for a uniform alloy (Fig. b) for given x. If we fix the temperature T, $f(x)$ can be regarded as a function of the composition parameter x. Let us consider the behavior of the function $f(x)$.

At the extrema of the function $f(x)$, x satisfies the equation

$$\frac{zV}{k_B T}(2x - 1) = g(x), \tag{2}$$

with a function $g(x)$. In addition to the solution $x = \frac{1}{2}$, this equation has other solutions when $T < T_{c0}$. Such solutions can be written as $x = \frac{1}{2} \pm \delta_0$ with $\delta_0 > 0$. Also, $f(x)$ has inflection points (solutions to $f''(x) = 0$) which can be written as $x = \frac{1}{2} \pm \delta_1$ with $\delta_1 > 0$.

[4] Find $g(x)$.

[5] At $T = T_{c0}$, the slopes on both sides of Eq. (2) coincide at $x = \frac{1}{2}$. Find T_{c0}.

[6] Draw $y = g(x)$ and $y = \frac{zV}{k_B T}(2x - 1)$ in a single graph for the cases of $T > T_{c0}$ and $T < T_{c0}$.

[7] Find δ_1.

[8] Draw $f(x)$ as a function of x in the case $T < T_{c0}$. In doing so, indicate the positions $x = \frac{1}{2} \pm \delta_0$ and $x = \frac{1}{2} \pm \delta_1$.

[9] δ_0 and δ_1 can be regarded as functions of the temperature T. Draw the curves $x = \frac{1}{2} \pm \delta_0(T)$ and the curves $x = \frac{1}{2} \pm \delta_1(T)$ in a single graph based on [7] and [8], with x and T being the horizontal and vertical axes, respectively.

Next, we investigate the phenomenon of phase separation in alloys. To this end, we study the stability of uniform alloys. Suppose that a uniform alloy M0 with $x = x_0$ (Fig. b) separates into a uniform alloy M1 with $x = x_1$ and a uniform alloy M2 with $x = x_2$ according to the ratio $s : (1-s)$ (Fig. c). Here we assume $x_1 < x_2$, which leads to $x_1 < x_0 < x_2$. We write the average free energy per site of the phase separated alloy as f^*. The uniform alloy with $x = x_0$ is energetically stable if

$$f(x_0) < f^* \tag{3}$$

holds.

[10] The number of A atoms in M0 is the same as the sum of those in M1 and M2. Express s in terms of x_0, x_1, and x_2.

[11] Express f^* in terms of x_0, x_1, x_2, and the function $f(x)$.

[12] A uniform alloy is stable against infinitesimal fluctuations of x, if Eq. (3) is satisfied for $x_1 = x_0 - \delta$ and $x_2 = x_0 + \delta$ with an infinitesimal δ. Prove that a uniform alloy with $x = x_0$ is stable against infinitesimal fluctuations of x if $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} > 0$.

[13] In the parameter space spanned by T and x, there appear three regions where (I) the uniform alloy is the most stable, (II) the uniform alloy is not the most stable but is stable against infinitesimal fluctuations of x (i.e., quasi-stable), and (III) the uniform alloy is unstable against infinitesimal fluctuations of x and undergoes phase separation. Show the regions I, II, and III in the graph obtained in [9].

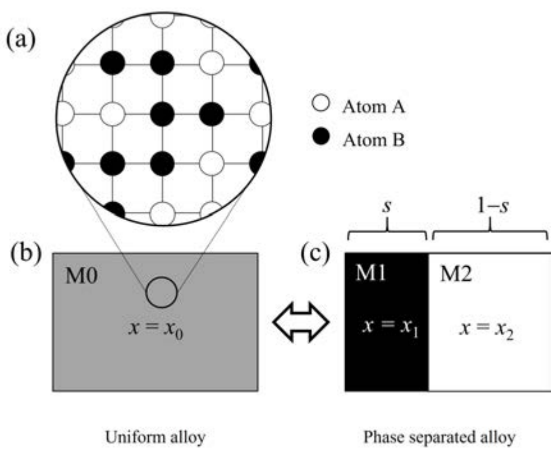

Figure : Schematics of a binary alloy. (a) Configuration of A and B atoms for a square lattice. Schematic pictures of (b) a uniform alloy and (c) its phase separation.

Problem 4

Answer the following questions regarding responses of a dielectric material exposed to an electromagnetic wave. The permittivity of vacuum and the imaginary unit are denoted by ϵ_0 and i, respectively.

[1] First, we consider the interaction of an atom with the electromagnetic wave. Suppose that the atom consists of an electron with mass m and electric charge $-q$ $(m, q > 0)$ which is bound to a spatially fixed nucleus. The atom is exposed to the spatially uniform electric field $\boldsymbol{E}(t) = \boldsymbol{E}_0 \exp(-i\omega t)$ with angular frequency ω $(\omega > 0)$. The vector $\boldsymbol{x}(t)$ is the displacement of the electron from its equilibrium position. There are three forces acting on the electron: the force from the electric field $\boldsymbol{E}(t)$, a restoring force $-m\omega_0^2 \boldsymbol{x}(t)$, and a velocity dependent damping force $-m\gamma \dfrac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}$ $(\omega_0, \gamma > 0)$. We use complex representations for the electron displacement and the electric field, where the real parts correspond to the physical quantities.

[1.1] Write down the equation of motion for the displacement $\boldsymbol{x}(t)$ of the electron.

[1.2] Assume the form $\boldsymbol{x}(t) = \boldsymbol{x}_0 \exp(-i\omega t)$ as a solution of [1.1]. Find $\boldsymbol{x}_0$.

[1.3] Now, we consider a dielectric that consists of N atoms per unit volume, where each atom is described by the model above. The electric polarization density $\boldsymbol{P}(t)$ induced in the dielectric is expressed as $\boldsymbol{P}(t) = -Nq\boldsymbol{x}(t)$. Define the complex susceptibility χ by $\boldsymbol{P}_0 = \epsilon_0 \chi \boldsymbol{E}_0$, where $\boldsymbol{P}_0$ is given by $\boldsymbol{P}(t) = \boldsymbol{P}_0 \exp(-i\omega t)$. We write $\chi = \chi_R + i\chi_I$, where χ_R and χ_I are real. Derive expressions for χ_R and χ_I.

[1.4] Draw the graph of χ_R as a function of ω, assuming $\gamma \ll \omega_0$. It is not necessary to calculate the extremal values of χ_R.

[2] As in Fig.1, we suppose that the region $z \leq 0$ is filled with the dielectric described in [1], and the region $z > 0$ is vacuum. We consider the response of the dielectric to an incident electromagnetic wave with an angular frequency ω, which is higher than the angular frequency ω_0. We approximate χ as a constant satisfying $-1 < \chi_R < 0$ and $\chi_I = 0$, and we write the electric polarization density $\boldsymbol{P}(\boldsymbol{r}, t) = \epsilon_0 \chi \boldsymbol{E}(\boldsymbol{r}, t)$ ($\chi = 0$ in vacuum) in terms of the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ at the position $\boldsymbol{r}$. The permeability of the dielectric is equal to μ_0, the permeability of vacuum. Neither true charges nor true currents exist at the interface and in the dielectric.

In the following, we consider a plane electromagnetic wave with the incident angle θ_0 $(0 < \theta_0 < \frac{\pi}{2})$ entering from the vacuum to the dielectric. Figure 1 shows the situation where there exists a transmitted wave in the dielectric. Suppose that the electric fields of the incident wave $\boldsymbol{E}_0(\boldsymbol{r}, t)$ and the reflected wave $\boldsymbol{E}_m(\boldsymbol{r}, t)$ in vacuum, and the electric field in the dielectric $\boldsymbol{E}_t(\boldsymbol{r}, t)$ are expressed as

$$\boldsymbol{E}_0(\boldsymbol{r}, t) = E_0 \boldsymbol{e}_y \exp\{i(\boldsymbol{K}_0 \cdot \boldsymbol{r} - \omega t)\},$$

$$\boldsymbol{E}_m(\boldsymbol{r}, t) = E_m \boldsymbol{e}_y \exp\{i(\boldsymbol{K}_m \cdot \boldsymbol{r} - \omega t)\},$$

$$\boldsymbol{E}_t(\boldsymbol{r}, t) = E_t \boldsymbol{e}_y \exp\{i(\boldsymbol{K}_t \cdot \boldsymbol{r} - \omega t)\},$$

where the wave vectors $\boldsymbol{K}_0, \boldsymbol{K}_m$, and $\boldsymbol{K}_t$ are

$$\boldsymbol{K}_0 = (k \sin\theta_0, 0, -k\cos\theta_0), \qquad \boldsymbol{K}_m = (K_{mx}, 0, K_{mz}), \qquad \boldsymbol{K}_t = (K_{tx}, 0, K_{tz}).$$

The vector $\boldsymbol{e}_y$ is a unit vector along the positive y-direction. Also, k is a positive real number, while E_0, E_m, and E_t are complex numbers in general.

Maxwell's equations in this system are written as

$$\nabla \cdot (\epsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)) = 0, \tag{1}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t), \tag{2}$$

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B}(\boldsymbol{r},t) = \frac{\partial}{\partial t}(\epsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)), \tag{3}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0, \tag{4}$$

with the magnetic flux density $\boldsymbol{B}(\boldsymbol{r},t)$.

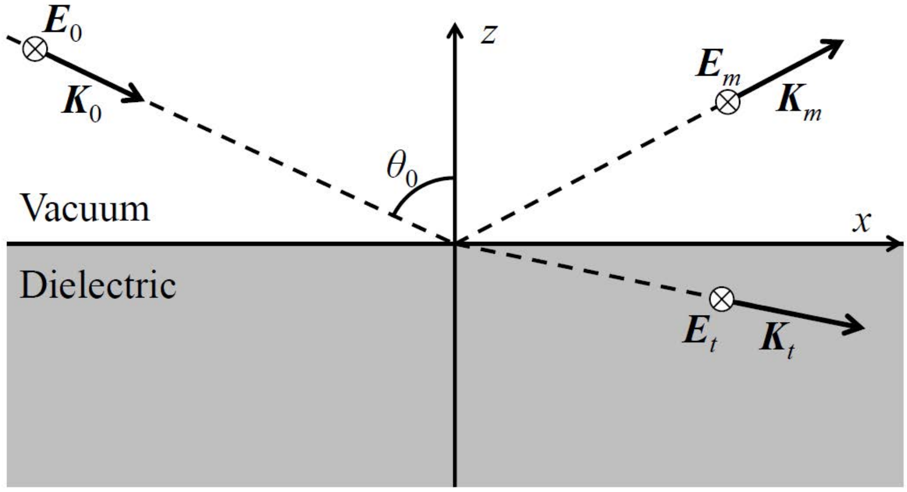

Figure 1

[2.1] Express the continuity condition for the y component of the electric field at $z = 0$ as a relation of $E_0 \exp(i\boldsymbol{K}_0 \cdot \boldsymbol{r})$, $E_m \exp(i\boldsymbol{K}_m \cdot \boldsymbol{r})$, and $E_t \exp(i\boldsymbol{K}_t \cdot \boldsymbol{r})$. You may derive this condition by using surface integration of Eq. (2) on a rectangle of infinitesimal size with the corners $(x, y \pm \Delta y, \pm\Delta z)$, where $0 < \Delta z \ll \Delta y$.

[2.2] Express K_{mx} and K_{tx} in terms of k and θ_0 from the condition obtained in [2.1].

[2.3] Write down the wave equations of $\boldsymbol{E}_0(\boldsymbol{r},t)$ and $\boldsymbol{E}_m(\boldsymbol{r},t)$ in vacuum, and of $\boldsymbol{E}_t(\boldsymbol{r},t)$ in the dielectric. From the wave equations, express $k^2, K_{mx}^2 + K_{mz}^2$, and $K_{tx}^2 + K_{tz}^2$ in terms of $\epsilon_0, \mu_0, \omega$, and χ. You may use the formula from vector analysis, $\nabla \times (\nabla \times \boldsymbol{C}(\boldsymbol{r},t)) = \nabla(\nabla \cdot \boldsymbol{C}(\boldsymbol{r},t)) - \nabla^2 \boldsymbol{C}(\boldsymbol{r},t)$.

[2.4] Express K_{mz} in terms of k and θ_0 using the results of [2.2] and [2.3].

[2.5] When the incident angle is larger than some angle θ_c, K_{tz} becomes purely imaginary, and the electric field $\boldsymbol{E}_t(\boldsymbol{r},t)$ in the dielectric decays along the z-negative direction. In this case, all of the energy of the incident wave is reflected. This phenomenon is called total reflection. Find θ_c using the results of [2.2] and [2.3]. Express K_{tz} in terms of k, θ_0, and χ in the two cases $\theta_0 > \theta_c$ and $\theta_0 < \theta_c$.

[2.6] Suppose that the incident angle θ_0 is larger than θ_c of [2.5]. Moreover, we define z_0 as the absolute value of z where the amplitude of the electric field in the dielectric becomes $1/e$ (e is the base of the natural logarithm) of its value at $z = 0$. Express z_0 in terms of k, θ_0, and χ. Draw the graph of z_0 as a function of $\cos^2\theta_0$.