Department of Applied Physics
Entrance Examination Booklet

Physics

(Answer the 4 Problems in this Booklet)

August 27 (Tuesday) 9:00 – 13:00, 2019

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the four problems in this booklet.
4. Use one answer sheet for each problem (four answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider a one-dimensional classical system of N particles of mass m that are interconnected with springs as shown in Figure 1. We label the particles with $n = 1, 2, \ldots, N-1, N$. The springs at the two ends are attached to the immovable walls. All the springs have the same spring constant k and are initially at their natural length as depicted in Figure 1. We denote by $x_n(t)$ the change in position of n-th particle at time t, as measured from the position in Figure 1. Neglecting friction with the floor, answer the following questions.

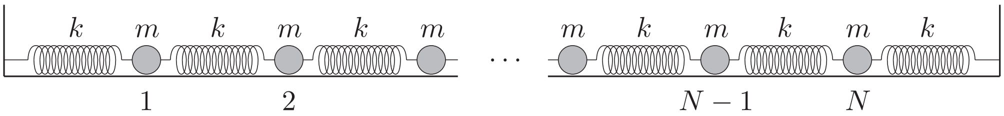

Figure 1

[1] Let us define $\boldsymbol{x}(t)$ as an N-dimensional column vector whose n-th component is given by $x_n(t)$.

[1.1] The equation of motion can accordingly be written as

$$m\frac{d^2}{dt^2}\boldsymbol{x}(t) = -K\boldsymbol{x}(t). \tag{1}$$

Find the N-dimensional square matrix K.

[1.2] Normalized eigenvectors $\boldsymbol{u}_\ell$ ($\ell = 1, 2, \ldots, N-1, N$) of the matrix K are given by

$$(\boldsymbol{u}_\ell)_n = N_\ell \sin(q_\ell n), \tag{2}$$

where $q_\ell = \dfrac{\pi\ell}{N+1}$ and $(\boldsymbol{u}_\ell)_n$ is the n-th component of the N-dimensional column vector $\boldsymbol{u}_\ell$ ($n = 1, 2, \ldots, N-1, N$). Obtain the eigenvalue of K associated with the eigenvector $\boldsymbol{u}_\ell$. Also, find the normalization factor N_ℓ. You may use the formula

$$\sum_{n=1}^{N+1} \cos\left(\frac{2\pi\ell n}{N+1}\right) = 0 \tag{3}$$

that holds for $\ell = 1, 2, \ldots, N-1, N$.

[1.3] We expand $\boldsymbol{x}(t)$ in terms of the eigenvectors $\boldsymbol{u}_\ell$ as $\boldsymbol{x}(t) = \displaystyle\sum_{\ell=1}^{N} \alpha_\ell(t)\boldsymbol{u}_\ell$. Show that the equation of motion for $\boldsymbol{x}(t)$ in Equation (1) leads to the following differential equation for $\alpha_\ell(t)$,

$$\frac{d^2}{dt^2}\alpha_\ell(t) = -\omega_\ell^2\alpha_\ell(t). \tag{4}$$

Given that $\omega_\ell > 0$, express the angular frequency ω_ℓ in terms of k, m, and q_ℓ.

[1.4] Draw a graph for the angular frequency ω_ℓ obtained in Question [1.3] as a function of q_ℓ.

[2] For the setup in Figure 1, an external force is applied to each particle individually. Let f_n be the force acting on the n-th particle and x_n be its corresponding displacement from the initial equilibrium position to the new one. With these, we construct the N-dimensional vectors $\boldsymbol{f}$ and $\boldsymbol{x}$, whose n-th components are f_n and x_n, respectively. We then expand these vectors as $\boldsymbol{x} = \sum_{\ell=1}^{N} \alpha_\ell \boldsymbol{u}_\ell$ and $\boldsymbol{f} = \sum_{\ell=1}^{N} \beta_\ell \boldsymbol{u}_\ell$ with $\boldsymbol{u}_\ell$ defined in Question [1.2].

[2.1] Express the ratio α_ℓ/β_ℓ in terms of m and ω_ℓ obtained in Question [1.3].

[2.2] Find ℓ that maximizes the ratio α_ℓ/β_ℓ. How does the maximum value of α_ℓ/β_ℓ behave as a function of N when $N \gg 1$?

[3] As schematically shown in Figure 2, an additional potential $\frac{1}{2}k_0 x_n(t)^2$ is applied to each particle in the initial setup in Figure 1. Here, k_0 (> 0) is a constant characterizing the strength of the potential.

[3.1] How does the matrix K in Equation (1) change in this situation? Explain how this additional potential affects the eigenvalues and eigenvectors of K.

[3.2] Describe the corresponding angular frequency ω_ℓ in terms of k, k_0, m, and q_ℓ given in Question [1.2]. Also, find the minimum value of the angular frequencies ω_ℓ in the limit $N \to \infty$.

[3.3] Draw a graph for the angular frequency ω_ℓ obtained in Question [3.2] as a function of q_ℓ.

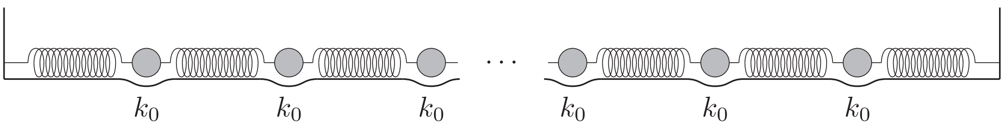

Figure 2

Problem 2

Consider a plane electromagnetic wave with the angular frequency ω and the wave vector $\boldsymbol{k}$ propagating in an optically anisotropic medium. $\boldsymbol{k}$ is a column vector $\boldsymbol{k} = (k_x, k_y, k_z)^{\mathrm{T}}$, where T denotes transposition. When the electromagnetic wave is not absorbed or scattered in the medium, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ of the plane wave in the medium can be expressed in the following complex notation,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}, \quad \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}_0 \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}. \tag{1}$$

Here, $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are constant complex column vectors independent of position $\boldsymbol{r} = (x, y, z)^{\mathrm{T}}$ and time t. In this medium, Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{2}$$

are satisfied. Here, $\boldsymbol{B}$ and $\boldsymbol{D}$ are the magnetic flux density and the electric flux density, respectively. The magnetic permeability of the medium is assumed to be equal to the vacuum permeability μ_0, and the relation $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ holds in the medium. This medium is a uniaxial crystal whose electric permittivity in the z direction is different from that in the other directions; $\boldsymbol{D}$ and $\boldsymbol{E}$ satisfy the relation $\boldsymbol{D} = \tilde{\varepsilon}\boldsymbol{E}$ with a matrix

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}. \tag{3}$$

Here, permittivities ε_1 and ε_2 satisfy $\varepsilon_1 > \varepsilon_2 > 0$.

[1] Given that the plane wave in Equations (1) satisfies Maxwell's equations (2), derive the following relation,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) + \omega^2 \mu_0 \tilde{\varepsilon} \boldsymbol{E}_0 = 0. \tag{4}$$

[2] Equation (4) can be transformed into the form $\tilde{X}\boldsymbol{E}_0 = 0$ with a matrix $\tilde{X}$. Derive an expression for $\tilde{X}$. You may use the formula $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ for the three-dimensional vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

[3] Assume that the wave vector of the plane wave is given by $\boldsymbol{k} = (0, k\sin\theta, k\cos\theta)^{\mathrm{T}}$ with $0 < \theta < \pi/2$ and $k > 0$. Show that Equation (4) has a solution for $\boldsymbol{E}_0$ other than a zero vector only when k is equal to one of the two values given by

$$k_1 = \omega\sqrt{\mu_0 \varepsilon_1}, \tag{5}$$

$$k_2 = \omega\sqrt{\frac{\mu_0 \varepsilon_1 \varepsilon_2}{\varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta}}. \tag{6}$$

[4] Find solutions for $\boldsymbol{E}_0$ corresponding to k_1 and k_2 in Question [3]. Here, we set $|\boldsymbol{E}_0| = E_0$.

Suppose that this uniaxial crystal is cut into a rectangular parallelepiped, as shown in Figure 1. The z axis of the crystal is inclined at an angle θ with respect to the normal direction of the side surface A of the crystal, and the surface A is perpendicular to the yz plane. Consider the case where a light beam with the angular frequency ω is normally incident on the surface A from the air. Since the wave vector of the light beam remains normal to the surface A after entering into the crystal, Figure 1 represents the situation considered in Question [3]. In this case, the light beam in the crystal is split into two beams propagating in different directions, one beam with only the x component of the electric field and the other beam with only the y and z components of the electric field. Here, we neglect reflection at the surfaces of the crystal and light dispersion in the crystal.

[5] In Figure 1, consider the light beam with only the x component of the electric field in the crystal. Which solution in Question [3], $k = k_1$ or $k = k_2$, corresponds to this beam? Also, show that this beam keeps propagating straight without changing the direction after entering into the crystal. Note that the propagation direction of a light beam is given by the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ and does not always coincide with the direction of the wave vector.

[6] In Figure 1, consider the light beam with only the y and z components of the electric field in the crystal. The propagation direction of this beam is tilted at an angle α after entering into the crystal. Express $\tan(\theta + \alpha)$ in terms of ε_1, ε_2, and θ. Then, express $\tan\alpha$ in terms of ε_1, ε_2, and θ.

[7] Consider a sheet of paper on which a character "Q" is printed as shown in Figure 2. The crystal in Figure 1 is placed in such a way that its surface A comes into contact with the surface of the paper. Suppose that we view the character printed on the paper from the opposite side of the surface A of the crystal. Illustrate what it looks like. Also, explain the reason using a diagram.

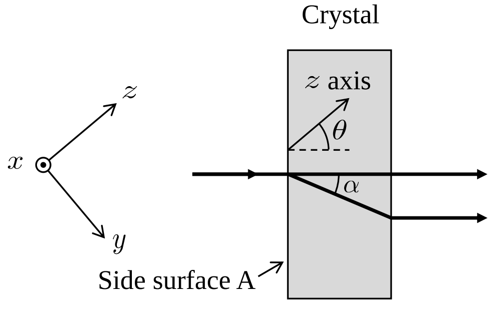

Figure 1

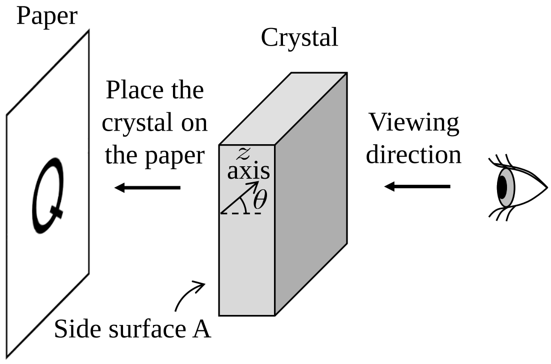

Figure 2

Problem 3

As a simple model for adsorption of atoms on a solid surface, consider a system where a monatomic ideal gas is in contact with an adsorbent lattice, consisting of an array of N_a adsorption sites (see Figure 1). The system obeys the Boltzmann statistics. The adsorption sites are independent of each other, and each adsorption site can take either two states with adsorbed atoms (adatoms): zero adatom with the energy 0 and one adatom with the energy $-\varepsilon$ ($\varepsilon > 0$). The mass of each atom is denoted by m. Internal degrees of freedom of the atoms are neglected. The entire system is in a thermal equilibrium state at temperature T and chemical potential μ. Let k_{B} be the Boltzmann constant, $\beta = 1/(k_{\mathrm{B}}T)$ the inverse temperature, and $\hbar$ the Planck constant divided by 2π.

[1] First, consider only the monatomic ideal gas. Show that the partition function of the ideal gas, consisting of N monatoms in a volume V, is given by

$$Z^{(\mathrm{g})}(V, \beta, N) = \frac{V^N}{N!}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3N/2}. \tag{1}$$

[2] Find the grand partition function $Z_{\mathrm{G}}^{(\mathrm{g})}(V, \beta, \mu) = \sum_{N=0}^{\infty} Z^{(\mathrm{g})}(V, \beta, N)e^{\beta\mu N}$ by using Equation (1).

[3] Pressure P of the ideal gas is given by $Z_{\mathrm{G}}^{(\mathrm{g})}$ as $P(\beta, \mu) = \dfrac{1}{\beta}\dfrac{\partial}{\partial V}\log Z_{\mathrm{G}}^{(\mathrm{g})}(V, \beta, \mu)$. Find $P(\beta, \mu)$ by applying the result in Question [2] to this relation.

[4] Next, consider the situation where the adsorbent lattice is in contact with this monatomic ideal gas. Find the grand partition function $\xi_{\mathrm{G}}^{(\mathrm{a})}$ for the states at a single adsorption site.

[5] The grand partition function of the entire adsorbent lattice is given by $Z_{\mathrm{G}}^{(\mathrm{a})} = (\xi_{\mathrm{G}}^{(\mathrm{a})})^{N_a}$. Using this, find the adatom density n_a (the total number of adsorbed atoms divided by N_a).

[6] Using the results obtained in Questions [3] and [5], express the adatom density n_a as a function of pressure P and temperature T.

[7] Plot n_a obtained in Question [6] as a function of pressure P at a constant temperature T. Also, plot n_a as a function of temperature T at a constant pressure P.

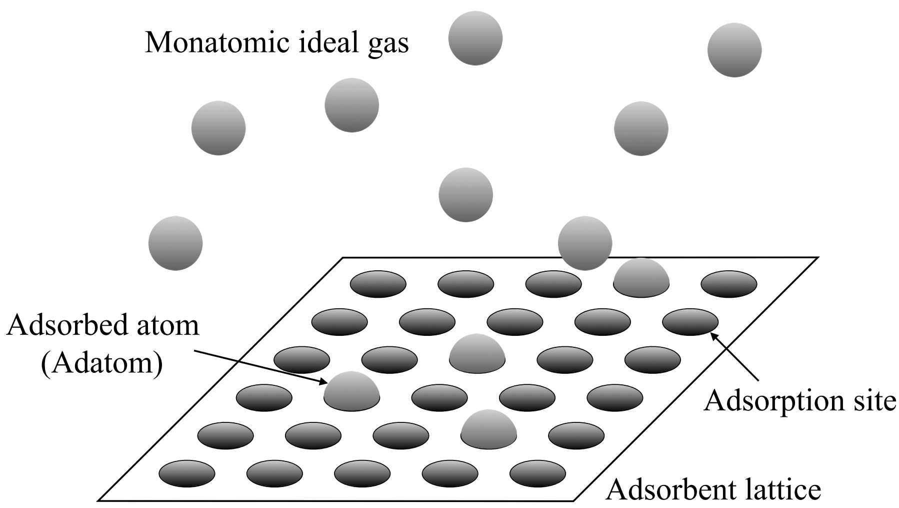
Monatomic ideal gas
Adsorbed atom
(Adatom)
Adsorption site
Adsorbent lattice

Figure 1

Problem 4

Consider a quantum-mechanical particle of mass m moving in two dimensions under the potential $(1/2)m\omega^2(\hat{x}^2+\hat{y}^2)$. Here, ω is a positive constant characterizing the potential strength, and $\hat{x}$ ($\hat{y}$) is the position operator along the x (y) axis. First, consider the motion in the x and y directions separately. Let $|n\rangle$ be the eigenstate of a particle of mass m moving along the x axis under the potential $(1/2)m\omega^2\hat{x}^2$, where the integer n $(=0,1,2,\dots)$ labels the states in the increasing order of energy. Let us introduce operators,

$$\hat{a}_x^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}_x\right),\quad \hat{a}_x=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}_x\right), \tag{1}$$

with $\hat{p}_x$ being the x component of the momentum operator, and $\hbar$ denoting the Planck constant divided by 2π. Accordingly, $\hat{a}_x^\dagger$, $\hat{a}_x$, and $|n\rangle$ satisfy the following relations:

$$[\hat{a}_x,\hat{a}_x^\dagger]=\hat{a}_x\hat{a}_x^\dagger-\hat{a}_x^\dagger\hat{a}_x=1, \tag{2}$$

$$\hat{a}_x^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle,\quad \hat{a}_x|n\rangle=\sqrt{n}|n-1\rangle\quad (n\neq 0),\quad \hat{a}_x|0\rangle=0. \tag{3}$$

Similarly, we introduce the operators for motion in the y direction as follows,

$$\hat{a}_y^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{y}-\frac{i}{m\omega}\hat{p}_y\right),\quad \hat{a}_y=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{y}+\frac{i}{m\omega}\hat{p}_y\right). \tag{4}$$

Ignoring the spin of the particle (except in Question [9]), answer the following questions.

[1] Describe the overall Hamiltonian $\hat{H}$ of the system using $\hat{a}_x^\dagger, \hat{a}_x, \hat{a}_y^\dagger$, and $\hat{a}_y$.

[2] Find the energy eigenvalues of $\hat{H}$. Find the degree of degeneracy in the N-th energy level $(N=1,2,3,\dots)$ from the lowest one.

[3] Express the z component of the angular momentum $\hat{l}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x$ in terms of $\hat{a}_x^\dagger, \hat{a}_x, \hat{a}_y^\dagger$, and $\hat{a}_y$. Also show that the angular momentum $\hat{l}_z$ is conserved.

Next, let us construct simultaneous eigenstates of the Hamiltonian $\hat{H}$ and the angular momentum $\hat{l}_z$. For this purpose, we introduce operators $\hat{b}^\dagger=C\hat{a}_x^\dagger+D\hat{a}_y^\dagger$ and $\hat{b}=C^*\hat{a}_x+D^*\hat{a}_y$. Here, C and D are complex coefficients which satisfy $|C|^2+|D|^2=1$, and C^* and D^* are their complex conjugates.

[4] Suppose that an operator $\hat{A}$ and a real number α satisfy

$$[\hat{l}_z,\hat{A}]=\alpha\hat{A}. \tag{5}$$

Let $|l_z\rangle$ be an eigenstate of $\hat{l}_z$ with the eigenvalue l_z. Show that $\hat{A}|l_z\rangle$ is also an eigenstate of $\hat{l}_z$ with the eigenvalue $l_z+\alpha$ when $\hat{A}|l_z\rangle\neq 0$.

[5] Find coefficients C and D, and the corresponding α so that the operator $\hat{b}^\dagger$ satisfies the condition for $\hat{A}$ in Equation (5). Choose C as a non-negative real number. Note that there exist two choices for $\hat{b}^\dagger$ and $\hat{b}$. We denote them by $\hat{b}_1^\dagger$ and $\hat{b}_1$, $\hat{b}_2^\dagger$ and $\hat{b}_2$ below.

[6] Calculate the commutation relations $[\hat{b}_1,\hat{b}_1^\dagger]$, $[\hat{b}_2,\hat{b}_2^\dagger]$, $[\hat{b}_1,\hat{b}_2^\dagger]$, and $[\hat{b}_2,\hat{b}_1^\dagger]$.

[7] Express the Hamiltonian $\hat{H}$ and the angular momentum $\hat{l}_z$ in terms of $\hat{b}_1^\dagger, \hat{b}_1, \hat{b}_2^\dagger$, and $\hat{b}_2$.

[8] Find all the eigenvalues of $\hat{l}_z$ for the eigenstates belonging to the N-th energy level ($N = 1, 2, 3, \ldots$).

[9] Suppose that the particle considered so far is an electron with spin 1/2. Let us introduce the spin-orbit interaction $\hat{H}_{\mathrm{so}} = \lambda \hat{l}_z \sigma_z$. Here, λ is the magnitude of the spin-orbit interaction, and $\sigma_z = \pm 1$ corresponds to the z component of spin. Find the energy levels of the system in this case.