Department of Applied Physics
Entrance Examination Booklet

Physics I

(Answer the 2 Problems in this Booklet)

September 1 (Tuesday) 9:30 – 11:30, 2015

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the two problems in this booklet.
4. Use one answer sheet for each problem (two answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider a simple pendulum, composed of a point mass m_1 suspended from the ceiling by a massless rigid rod of length l_1 (see Figure 1). The pendulum can freely swing under the influence of acceleration of gravity g in the xy plane (the x-axis is along the horizontal direction and the y-axis is along the vertical direction, pointing towards the ceiling).

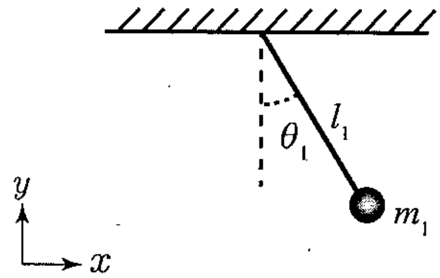

Figure 1

[1] Derive the equation of motion for this pendulum. Assuming that the amplitude of oscillations is small, prove that the pendulum obeys simple harmonic motion with an angular frequency of $\omega_0 = \sqrt{g/l_1}$.

[2] Qualitatively explain why the angular frequency ω_0 is independent of m_1. Assuming that the harmonic oscillations are started by a certain impulsive force, applied to m_1 at its equilibrium position, qualitatively explain how the oscillations are affected by the value of m_1.

Next, let us consider that the mass m_1 in Figure 1 is connected to another point mass m_2 with a massless rigid rod of length l_2. This rod is allowed to freely swing about the point mass m_1 in the xy plane (see Figure 2).

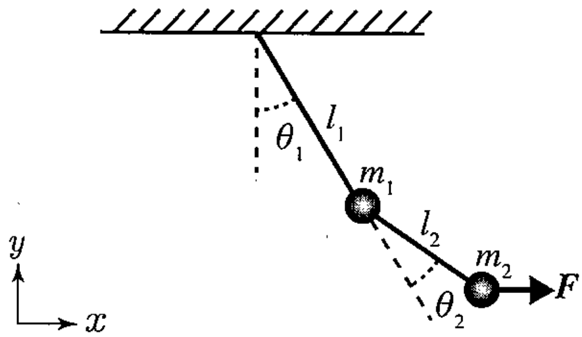

Figure 2

[3] As shown in Figure 2, a constant force $\boldsymbol{F} = (F_x, 0)$ applied to the point mass m_2 along the x direction brings the system into a new state of equilibrium. Find θ_1 and θ_2 in this state. Note that these angles are not necessarily small.

[4] If the force $\boldsymbol{F}$ described in [3] is removed, both of the masses move from their equilibrium positions and start oscillating. Taking $\boldsymbol{F} = \mathbf{0}$ into account, derive the Lagrangian of this oscillating system. Suppose that $m_1 = m_2 = m$ and $l_1 = l_2 = l$ (see Figure 3).

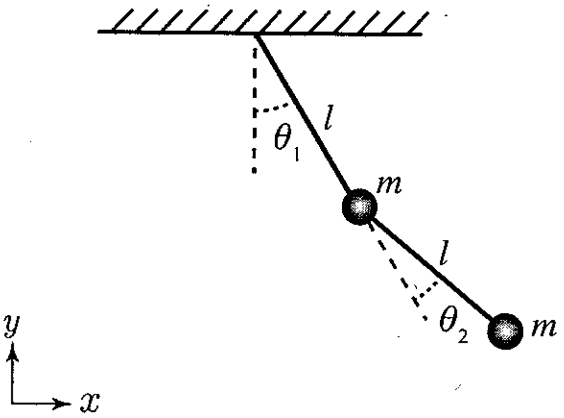

Figure 3

[5] Using the Lagrangian derived in [4] and assuming that the swing angles θ_1 and θ_2 and their corresponding angular velocities $d\theta_1/dt$ and $d\theta_2/dt$ (t is time) are small, find the equations of motion.

[6] Using the equations of motion obtained in [5], show that the system has two normal modes with angular frequencies $\omega = \omega_0\sqrt{2 \pm \sqrt{2}}$.

Problem 2

Consider a circular ring made of a thin conducting wire with radius a, mass m, and electric resistance of one full turn R, subject to a magnetic field with the magnetic flux density vector (B_x, B_y, B_z), where

$$B_x = B_0 \alpha z,\ B_y = 0,\ B_z = B_0(1 + \alpha x).$$

Here B_0 and α are positive constants. As shown in Figure 1, the ring is placed in the xy plane ($z = 0$); its center is always on the x axis, and the ring can move along the x axis without any rotation and deformation. The x coordinate of the center of the ring is denoted by X. The ring is charge neutral, and the effects of friction and self-inductance of the ring are negligible. The current flows uniformly along the ring.

[1] Find the magnetic flux enclosed by the ring, Φ, as a function of X.

[2] By applying a constant force along the x direction, the ring moves with a constant velocity $V(> 0)$, and consequently, a constant current I is induced along the ring. Obtain I. In Figure 1, the sign of the current is defined to be positive if it circulates counterclockwise.

[3] Obtain the magnitude of the applied force F.

Next, we consider a case wherein the magnetic field changes as a function of time t. The z component of the magnetic flux density vector is given by

$$B_z = B_0 b(t)(1 + \alpha x),$$

where $b(t)$ is a function of time, expressed as

$$b(t) = \begin{cases} \dfrac{t}{\tau} & (0 \leqq t < \tau) \\ 1 & (\tau \leqq t) \end{cases}.$$

Here, $\tau > 0$. At $t = 0$, the ring is at rest and its center is at the origin.

[4] Show that the equation of motion of the ring can be expressed in the form

$$\alpha \frac{d^2 X}{dt^2} = -\lambda b(t) \frac{d}{dt}\left[b(t)(1 + \alpha X)\right], \tag{1}$$

and obtain λ in Equation (1).

[5] Consider the case $0 \leqq t < \tau$. To solve Equation (1), one can use the following power-series expansion,

$$X = \sum_{k=0}^{\infty} a_k t^k.$$

Show that $a_0 = a_1 = a_2 = 0$.

If τ is much smaller than $1/\lambda$, the solution of Equation (1) can be approximated by $X = a_3 t^3$ for $0 \leqq t < \tau$. Under this condition, answer the following questions.

[6] Obtain a_3 and the velocity of the ring, V_0, at $t = \tau$.

[7] When $t \to \infty$, the ring will stop. Obtain the final position of the center of the ring.

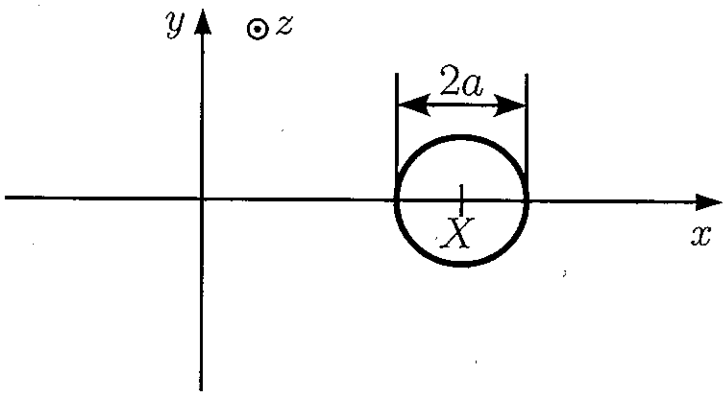

Figure 1

Department of Applied Physics
Entrance Examination Booklet

Physics II

(Answer 3 Problems among the 4 Problems in this Booklet)

September 1 (Tuesday) 13:00 – 16:00, 2015

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
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Examinee number	No.

Write down your examinee number above

Problem 1

Consider a one-dimensional harmonic oscillator with mass m and angular frequency ω. The Hamiltonian of the system is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

where $\hat{x}$ is the position operator and $\hat{p}$ is the momentum operator. The eigenenergy of the Hamiltonian is given by $E_n = (n + 1/2)\hbar\omega$, where n is a non-negative integer and $\hbar$ is the Planck constant divided by 2π. We introduce a creation operator $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{x}\sqrt{\frac{m\omega}{\hbar}} - i\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right)$ and an annihilation operator $\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{x}\sqrt{\frac{m\omega}{\hbar}} + i\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right)$, which satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$. With these, the Hamiltonian is rewritten as $\hat{H} = \hbar\omega(\hat{N} + 1/2)$ where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator.

[1] Find the commutation relation $[\hat{N}, \hat{a}]$. For any non-negative integer n, prove that an appropriate eigenstate $|n\rangle$ satisfying $\hat{N}|n\rangle = n|n\rangle$ fulfills the equation $\hat{a}|n+1\rangle = \sqrt{n+1}\,|n\rangle$. Show also $\hat{a}|0\rangle = 0$. Recall that energy levels are non-degenerate for a one-dimensional harmonic oscillator.

[2] For an arbitrary complex number α, prove that the state

$$|\Psi(\alpha)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is a normalized eigenstate of the annihilation operator $\hat{a}$. Find the eigenvalue of $\hat{a}$ for the state.

[3] Find the energy expectation value $\left\langle \hat{H} \right\rangle$ of the state $|\Psi(\alpha)\rangle$.

[4] Find the position and momentum expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$, and their respective root-mean-square deviations $\Delta x = \sqrt{\left\langle (\hat{x} - \langle \hat{x} \rangle)^2 \right\rangle}$ and $\Delta p = \sqrt{\left\langle (\hat{p} - \langle \hat{p} \rangle)^2 \right\rangle}$ for the state $|\Psi(\alpha)\rangle$. Calculate the product $\Delta x \Delta p$.

[5] Assume that the oscillator is in the state $|\Psi(\alpha_0)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle$ at time $t = 0$. We take $\alpha_0 = Ae^{i\theta}$ with A and θ being positive and real values. Show that the oscillator's state is given in the form $e^{-i\omega t/2}|\Psi(\alpha(t))\rangle$ at $t (> 0)$. Express $\alpha(t)$ in terms of A, θ, ω and t.

[6] For the oscillator's state derived in [5], find the expectation values of the position $\langle \hat{x} \rangle_t$ and the momentum $\langle \hat{p} \rangle_t$ at time t. This state is referred to as a quasi-classical state. Explain the reason by considering the behavior of the state for $A \gg 1$.

Problem 2

Consider the thermal changes in one mole of a gas undergoing volume expansion. Let P be the pressure, V the volume, R the gas constant, T the thermodynamic temperature, U the internal energy, $H = U + PV$ the enthalpy, S the entropy, $C_P = \left(\frac{\partial H}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P$ the specific heat at constant pressure, and $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P > 0$ the thermal expansion rate. Answer the following questions. Note that you may use the Maxwell relation $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$.

[1] Let us consider the quasi-static adiabatic expansion. This is a reversible isentropic (constant-entropy) adiabatic process, for which we define the rate of changes in the temperature with respect to the pressure as $\left(\frac{\partial T}{\partial P}\right)_S$. Show that this can be expressed as $\left(\frac{\partial T}{\partial P}\right)_S = \frac{TV\alpha}{C_P}$, and find α for an ideal gas obeying the equation $PV = RT$. Suppose that S is a function of P and T, and dS can be expressed as $dS = \left(\frac{\partial S}{\partial P}\right)_T dP + \left(\frac{\partial S}{\partial T}\right)_P dT$.

[2] Let us now consider the Joule-Thomson expansion, which is an irreversible adiabatic process. Under this expansion, the gas is passed from the high pressure (P_1) region at left to the low pressure (P_2) region at right through a porous plug as shown in Figure 1. By using piston 1 and piston 2, the gas flow through the porous plug is regulated such that both P_1 and P_2 remain constant. Accordingly, the volume and the temperature of the gas change from the initial values V_1 and T_1 to the final values V_2 and T_2, respectively, without any heat exchange between the gas and the surroundings. Show that such a process is isenthalpic (constant-enthalpy).

[3] For the isenthalpic process described in [2], the rate of changes in the temperature after a small change in the pressure is given by the Joule-Thomson coefficient $\left(\frac{\partial T}{\partial P}\right)_H$. Express this in terms of C_P, T, V and α. By comparing this to the coefficient $\left(\frac{\partial T}{\partial P}\right)_S$ deduced in [1], show that the relation $\left(\frac{\partial T}{\partial P}\right)_H < \left(\frac{\partial T}{\partial P}\right)_S$ holds. Finally, find the Joule-Thomson coefficient for an ideal gas. Suppose that H is a function of P and T.

Next, let us consider the change in the temperature of a real gas upon expansion. Suppose that the gas obeys the following equation,

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT, \qquad (1)$$

where a and b are positive constants and correspond to the contributions from the intermolecular attractive force and the excluded volume, respectively. Assuming $V \gg b$ and $T \geqq T_B \left(\equiv \frac{a}{Rb}\right)$ (Boyle temperature), answer the following questions.

[4] We first consider the deviation from the ideal gas. Expanding Equation (1) as power series in $\left(\frac{b}{V}\right)$ up to the first order, PV can be approximated as $PV \approx RT\left\{1 + B(T)\left(\frac{b}{V}\right)\right\}$. Express $B(T)$ in terms of a, b, R and T. Discuss the meaning of the Boyle temperature T_B.

[5] Next, let us consider the temperature variation upon the Joule-Thomson expansion. Using the approximation $V \approx \frac{RT}{P} + B(T)b$, derive an expression for the Joule-Thomson coefficient in

terms of C_P, a, b, R and T. Find the inversion temperature T_{inv} at which the Joule-Thomson coefficient changes its sign.

[6] When $T > T_{\text{inv}}$, the Joule-Thomson coefficient becomes negative, meaning that the gas heats up upon volume expansion. This is due to the fact that the contribution of the excluded volume b dominates that of a arising from the intermolecular attractive force. Assuming that the contribution of a is negligible, explain why the temperature increases.

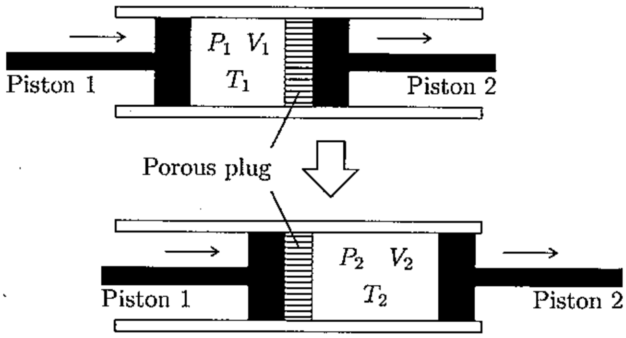

Figure 1

Problem 3

Consider the propagation of an electromagnetic wave in an isotropic medium. Let the permittivity and the permeability in vacuum be ε_0 and μ_0, respectively.

Suppose that the electromagnetic wave possesses the electric field vector,

$$\boldsymbol{E} = (E, 0, 0), \tag{1}$$

and

$$E = E_0 \exp\left[i\left(kz - \omega t\right)\right], \tag{2}$$

where k is the wave number, ω is the angular frequency, and E_0 is a constant. Using Maxwell's equations, we can derive the following equation:

$$\nabla^2 E - \varepsilon\mu_0 \frac{\partial^2 E}{\partial t^2} = 0. \tag{3}$$

Here, the permittivity of the medium, ε, is a function of ω. We assume that the force exerted to the charged particles in the medium by the magnetic field component of the electromagnetic wave is negligible and set the permeability of the medium to be equal to μ_0. In the following, energy dissipation can be ignored.

We first consider the propagation of the electromagnetic wave in a dielectric. Here, we regard the dielectric as a system composed of an ensemble of independent harmonic oscillators uniformly distributed with density N. Each harmonic oscillator consists of an electron, with mass m and electric charge $-e$, bound to a spatially fixed nuclei. It has the resonant angular frequency ω_0. Let $\boldsymbol{x}$ be the displacement vector of the electron and assume $\boldsymbol{x} = \boldsymbol{0}$ at equilibrium in the absence of an external electric field. Suppose the motion of the electron obeys classical mechanics.

[1] Find the equation of motion for the displacement vector, $\boldsymbol{x}$, of the electron.

[2] Given that the macroscopic polarization of the dielectric, $\boldsymbol{P}$, is solely produced by the harmonic oscillators described by the equation of motion in [1], derive the differential equation that relates the induced polarization, $\boldsymbol{P}$, to the electric field component, $\boldsymbol{E}$, of the electromagnetic wave.

[3] Under the electric field $\boldsymbol{E}$ in Equation (1), the induced polarization $\boldsymbol{P}$ in the dielectric is also written as

$$\boldsymbol{P} = (P, 0, 0). \tag{4}$$

It oscillates at the same angular frequency as $\boldsymbol{E}$ does. Using the result obtained in [2], derive the relation between E and P.

[4] Obtain the relationship between k and ω of the electromagnetic wave in the dielectric.

[5] Express the real and imaginary parts of k as a function of ω, and plot them schematically.

Next, we consider the propagation of an electromagnetic wave in a metal. Suppose that in the metal, free electrons with mass m are uniformly distributed with density n_e.

[6] Obtain the equation of motion for such electrons subject to the electric field component of the electromagnetic wave.

[7] For the electromagnetic wave in the metal, express the real and imaginary parts of k as a function of ω, and plot them schematically.

[8] Based on the results obtained in [5] and [7], explain the difference between the optical properties of dielectrics and metals in the visible light regime. Note that the angular frequency of the visible light falls into the low-frequency limit in most materials.

Problem 4

Consider a semiconductor under a uniform static magnetic field applied along the z direction. Assume that the motion of an electron can be described by the following semiclassical equations:

$$\frac{dx}{dt} = v_x = \frac{1}{\hbar}\frac{\partial \epsilon(\boldsymbol{k})}{\partial k_x}, \quad \frac{dy}{dt} = v_y = \frac{1}{\hbar}\frac{\partial \epsilon(\boldsymbol{k})}{\partial k_y}, \quad \frac{dz}{dt} = v_z = \frac{1}{\hbar}\frac{\partial \epsilon(\boldsymbol{k})}{\partial k_z}, \tag{1}$$

$$\hbar\frac{d\boldsymbol{k}}{dt} = -e\left(\boldsymbol{v} \times \boldsymbol{B}\right). \tag{2}$$

Here, $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{v} = (v_x, v_y, v_z)$, and $\boldsymbol{k} = (k_x, k_y, k_z)$ are the position, group velocity, and wave vector of the electron, respectively, at time t. $\epsilon(\boldsymbol{k})$ corresponds to the energy dispersion relation. Let $\boldsymbol{B} = (0, 0, B)$ be the magnetic flux density ($B > 0$), e the elementary charge ($e > 0$), and $\hbar$ the Planck constant divided by 2π. Ignore the electron's spin degrees of freedom and the quantization of the electron's orbit under the magnetic field.

[1] Let us assume that under the magnetic field the motion of an electron in the conduction band can be described by using the following dispersion relation,

$$\epsilon(\boldsymbol{k}) = \frac{\hbar^2 k_x^2}{2m_1} + \frac{\hbar^2 k_y^2}{2m_2} + \frac{\hbar^2 k_z^2}{2m_3}, \tag{3}$$

for Equations (1) and (2). The nonequivalence of the effective masses m_1, m_2 and m_3 reflects the anisotropy of the band. Here we assume $m_3 > m_2 > m_1 > 0$. Let $\boldsymbol{r}_0 = (0, 0, 0)$ and $\boldsymbol{k}_0 = (k_{0x}, 0, k_{0z})$ be the position and the wave vector of the electron at $t = 0$, respectively, with $k_{0x} > 0$ and $k_{0z} > 0$. Answer the following questions.

[1.1] Derive the density of states per unit volume as a function of energy ϵ, $\rho(\epsilon)$, for the band given by Equation (3). Schematically draw $\rho(\epsilon)$.

[1.2] Let $k_x = k_{0x}\cos(\omega_c t)$ be the solution for k_x obtained from Equations (1) - (3). Derive ω_c and the solutions for k_y and k_z.

[1.3] Draw the trajectory of the electron's wave vector $\boldsymbol{k}$ by projecting it onto the k_x-k_y plane. Show the direction of the time evolution by arrows.

[1.4] Describe the characteristics of the trajectory related to the position $\boldsymbol{r}$ of the electron. Draw the trajectory by projecting it onto the x-y plane. Show the direction of the time evolution by arrows.

[2] Consider the motion of an electron obeying the linear band dispersion (see Figure 1) in a two-dimensional crystal fixed in the xy plane. Taking the band crossing point as the origin of the energy, $\epsilon(\boldsymbol{k})$ can be described by the following equation

$$\epsilon(\boldsymbol{k}) = \pm\gamma\sqrt{k_x^2 + k_y^2}, \tag{4}$$

where γ is a positive constant. Let $\boldsymbol{r}_0 = (0, 0, 0)$ and $\boldsymbol{k}_0 = (k_0, 0, 0)$ be the position and the wave vector of the electron at $t = 0$, respectively, with $k_0 > 0$. Answer the following questions.

[2.1] Derive the density of states per unit area, $\rho(\epsilon)$, for the band given by Equation (4). Schematically draw $\rho(\epsilon)$.

[2.2] Let $k_x = k_0\cos(\omega_c t)$ be the solution for k_x obtained from Equations (1), (2) and (4). Derive ω_c.

[2.3] Draw the trajectories of the wave vector $\boldsymbol{k}$ and position $\boldsymbol{r}$ of the electron in the k_x-k_y and x-y planes, respectively. Show the directions of their time evolutions by arrows. Answer for both branches of $\epsilon(\boldsymbol{k}) > 0$ and $\epsilon(\boldsymbol{k}) < 0$.

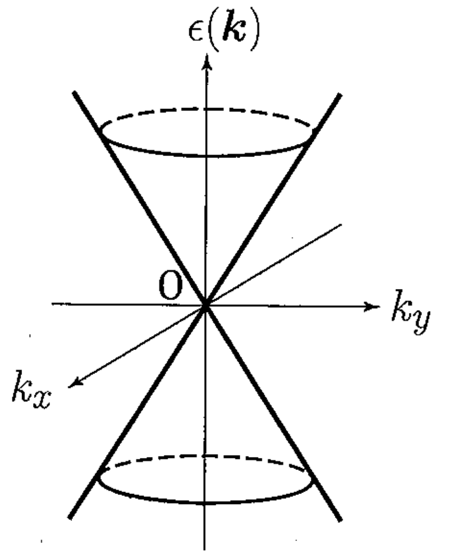

Figure 1