Department of Applied Physics
Entrance Examination Booklet

Physics I

(Answer the 2 Problems in this Booklet)

August 27 (Tuesday) 9:30 – 11:30, 2013

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the two problems in this booklet.
4. Use one answer sheet for each problem (two answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider a two-body problem with two point masses. Suppose that one of the point masses is at position $\boldsymbol{r}_1$ with mass of m_1 and the other is at position $\boldsymbol{r}_2$ with mass of m_2 subject to an interaction potential $U(\boldsymbol{r}_2 - \boldsymbol{r}_1)$. The Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\boldsymbol{r}_1}^2 + \frac{1}{2} m_2 \dot{\boldsymbol{r}_2}^2 - U(\boldsymbol{r}_2 - \boldsymbol{r}_1). \tag{1}$$

Here and hereafter t represents time and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ for a variable x.

[1] Show that the Lagrangian in Eq. (1) can be rewritten as

$$\mathcal{L} = \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(\boldsymbol{r}) \tag{2}$$

by using the center of mass coordinates $\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}$, the relative position coordinates $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$.

Express M and μ using m_1 and m_2.

From Eq. (2) it is clear that the above motion of the two point masses can be separated into two independent motions; a linear uniform motion of a point mass with mass of M (center of mass motion) and a motion of a point mass with mass of μ under the potential $U(\boldsymbol{r})$ (relative motion). Henceforth we shall focus only on the relative motion determined by the Lagrangian,

$$L = \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(\boldsymbol{r}). \tag{3}$$

Suppose that the potential $U(\boldsymbol{r})$ has spherical symmetry, that is, it depends only on $|\boldsymbol{r}| = r$. In this case the motion occurs in a plane. Let this plane be the xy-plane with the coordinates $\boldsymbol{r} = (r\cos\phi, r\sin\phi, 0)$.

[2] Show that the Lagrangian in Eq. (3) can be written as

$$L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r). \tag{4}$$

[3] From the Lagrangian L in Eq. (4) Lagrange's equation of motion for ϕ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}(\mu r^2 \dot{\phi}) = 0. \tag{5}$$

This indicates the conservation of the angular momentum $l = \mu r^2 \dot{\phi}$.

Show that Lagrange's equation of motion for r can be given by

$$\mu \ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}\right) \tag{6}$$

with the angular momentum l.

Explain the physical meaning of this equation.

[4] Multiply both sides of Eq. (6) by $\dot{r}$, and show that the energy $E = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}$ is conserved.

[5] The above results show that the angular momentum vector $\boldsymbol{l} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}} = (0, 0, l)$ and the energy $E = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}$ are conserved quantities for the motion determined by the Lagrangian L in Eq. (3).

For the case of the potential $U(r) = -\frac{k}{r}$ with a positive constant k, there is yet another conserved vector, the Laplace-Runge-Lenz vector

$$\boldsymbol{A} = \mu\dot{\boldsymbol{r}} \times \boldsymbol{l} - \mu k \frac{\boldsymbol{r}}{r} . \tag{7}$$

Show that $\boldsymbol{A}$ is perpendicular to $\boldsymbol{l}$.

Finally, derive the orbit equation

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + \frac{A}{\mu k}\cos\alpha\right) \tag{8}$$

by calculating $\boldsymbol{A} \cdot \boldsymbol{r} = Ar\cos\alpha$, where α is defined as an angle formed by $\boldsymbol{A}$ and $\boldsymbol{r}$ and $A = |\boldsymbol{A}|$.

Problem 2

In the following, use ϵ_0 and μ_0 for the permittivity and permeability of vacuum, respectively. Write down also the process of your calculations in the answer sheets.

[1] Consider the following situations where electric current generates a magnetic field.

[1.1] First, a ring current with radius a flows in the xy-plane as shown in Fig. 1. The center of the ring current is at the origin. The magnitude of the ring current is I. Calculate the magnitude of magnetic flux density at the point $(0, 0, z)$ and describe its direction.

[1.2] A solenoid with radius a, length l, and number of turns N is placed along the z-axis as shown in Fig. 2. Its center is at the origin. Calculate the magnitude of the magnetic flux density at the origin when a current with the magnitude I is applied to this solenoid.

[1.3] By assuming $l \gg a$, determine the self-inductance L of this solenoid. Here, assume the magnetic flux density inside the solenoid is uniform and equal to that at the origin.

[2] Consider a capacitor consisting of a pair of parallel conductor disks as shown in Fig. 3. The radius and the distance between the disks are b and $d(\ll b)$, respectively. Now, an ac voltage of $v(t) = v_0 \sin \omega t$ (where v_0 and ω are the amplitude and the angular frequency of $v(t)$, respectively) is applied to this capacitor. Find the electric field and magnetic field between the disks as functions of time t (also write down the directions of magnetic field). Here, use $r(< b)$ for the distance from the central axis of the disks.

[3] Consider a circuit including the solenoid and the capacitor (with capacitance C) used in questions [1.3] and [2], respectively (see Fig. 4). A switch is first connected to S1. After a certain period of time at which the ammeter indicates zero, the switch is connected to S2 (referred to as the initialized state). Ignore the resistance of the circuit. Assume that the distance between the solenoid and the capacitor is large.

[3.1] At time $t = 0$, the switch is connected to S3 (the solenoid and the capacitor are then connected). Explain quantitatively the time t dependence ($t > 0$) of the current flowing in the circuit by writing a circuit equation.

[3.2] Under the situation described in question [3.1], show the t dependence of the energy stored in the solenoid and the capacitor, respectively, in one graph. Explain the physical meaning of the dependence.

[3.3] Restarting from the initialized state, the distance between the disks of the capacitor is extended to $2d$. Then, the switch is connected to S3 at $t = 0$. In this situation, consider again plotting the t dependences of the energies as in question [3.2]. If there are quantitative differences in the t dependences of energies between these situations, write them down and explain why they arise.

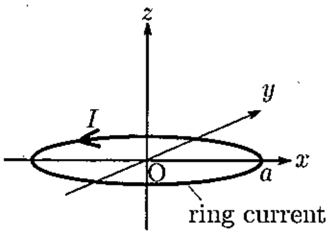


Fig. 1

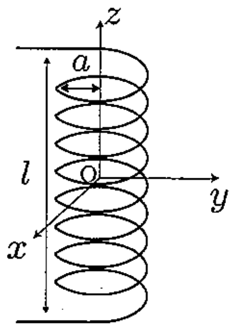


Fig. 2

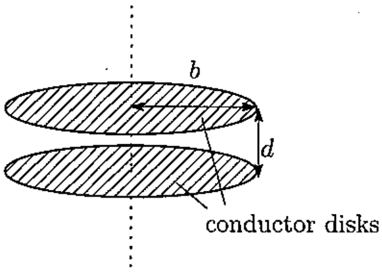


Fig. 3

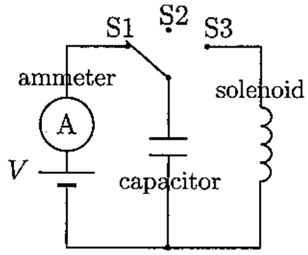


Fig. 4

Department of Applied Physics
Entrance Examination Booklet

Physics II

(Answer 3 Problems among the 4 Problems in this Booklet)

August 27 (Tuesday) 13:00 – 16:00, 2013

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

In the presence of an electric field, mobile electrons are subject to a spin-orbit interaction given by

$$H_{\mathrm{SO}} = \frac{\gamma}{\hbar}(\boldsymbol{p} \times \boldsymbol{E}) \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{p} = (p_x, p_y, p_z)$ is the momentum, γ is a negative constant, $\hbar$ is the Planck constant divided by 2π, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The above expression means that electrons are subject to an effective magnetic field, $\boldsymbol{B}_{\mathrm{SO}} = a(\boldsymbol{p} \times \boldsymbol{E})$ with a being a negative constant, due to the spin-orbit interaction.

[1] In the matrix representation with the basis of spin up and spin down along the z-direction, the Pauli matrices are written as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates of σ_z with eigenvalues $+1$ and -1 are given by

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Use the basis of α and β to determine the eigenvalue and eigenfunction for σ_x and σ_y, respectively.

[2] Consider the case where electrons moving in one-dimension along the x-axis are subject to a uniform static electric field along the z-direction, $\boldsymbol{E} = (0, 0, E)$ with $E > 0$.

[2.1] Show that the Hamiltonian can be written as

$$H = \frac{p_x^2}{2m} - \left(\frac{\gamma E}{\hbar}\right) p_x \sigma_y,$$

assuming that the effects of the electric field other than the spin-orbit interaction can be neglected. Here m is the electron mass.

[2.2] The eigenfunction ψ can be written as $\psi = \phi(x)\chi$ with the orbital wave function $\phi(x)$ and the spin wave function χ. Use this relation to calculate the eigenenergy and eigenstate of the Hamiltonian in [2.1]. Note that the orbital wave function is expressed by a plane wave function, $\phi(x) = e^{ikx}$ where k is the wave number. Plot schematically the energy dispersion relations for the calculated eigenenergies, i.e. energy versus k. Describe how the spin orientation changes depending on the wave number k along the energy dispersion curves.

[2.3] Consider that a uniform static magnetic field, $\boldsymbol{B} = (B, 0, 0)$ with $B > 0$ is further applied in addition to the static electric field $\boldsymbol{E}$. The electrons are then subject to a magnetic interaction or Zeeman effect given by

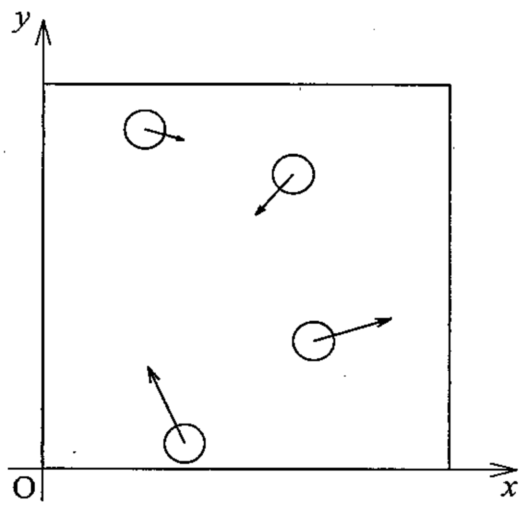
$$H_{\mathrm{Z}} = \frac{1}{2} g \mu_{\mathrm{B}} \boldsymbol{B} \cdot \boldsymbol{\sigma},$$

where g (> 0) is the Landé g-factor and μ_{B} is the Bohr magneton. Write down the Hamiltonian for the electrons and calculate the eigenenergies by diagonalizing this Hamiltonian. Note that the effects of the electric field other than the spin-orbit interaction can be neglected. Plot schematically the energy dispersion relations for the calculated eigenenergies and describe how the spin orientation changes depending on the wave number k along the dispersion curves.

Problem 2

N hard-disk particles are moving without friction and without dissipation in a square box, where N is an integer larger than or equal to two. Collisions between the particles, and between the particles and the walls of the box, are elastic. Ignore the particles's rotational degrees of freedom. The box is fixed in position and it does not move when the particles collide with walls. The total kinetic energy of this system is denoted by E, where E is positive.

The x-axis is defined along the lower horizontal wall of the box, and the y-axis along the other wall perpendicular to the x-axis (see the figure below). Each particle is identified by a number from 1 to N, and velocity of particle k ($1 \leq k \leq N$) is denoted by $\boldsymbol{v}_k = (v_{x,k},\ v_{y,k})$.



When we observe the system over a time period much longer than mean particle-particle collision time, the system obeys the microcanonical distribution. In the following, derive the canonical distribution from the microcanonical one.

Firstly, consider the case of $N = 2$.

[1] In a space made of four components of the velocity vectors of two particles, ($v_{x,1}$, $v_{y,1}$, $v_{x,2}$, $v_{y,2}$), show that the microcanonical distribution implies a uniform distribution on a surface of a four dimensional sphere, and determine its radius R.

In d (≥ 2) dimensional space, a set of points whose distances from a given point are less than or equal to a given r (> 0) is called a d dimensional sphere with radius r. The surface volume of the d dimensional sphere is denoted by $S_d(r)$, for example, $S_2(r)$, $S_3(r)$ and $S_4(r)$ measure circumference, surface area and volume, respectively.

[2] Show that the following expressions represent $S_d(r)$ for $d = 2$, 3 and 4, respectively:

$$\begin{aligned}
S_2(r) &= \int_{-r}^{r} \mathrm{d}q_1 \, \frac{2r}{\sqrt{r^2 - q_1^2}}, \\
S_3(r) &= \int_{-r}^{r} \mathrm{d}q_1 \int_{-\sqrt{r^2-q_1^2}}^{\sqrt{r^2-q_1^2}} \mathrm{d}q_2 \, \frac{2r}{\sqrt{r^2 - q_1^2 - q_2^2}}, \\
S_4(r) &= \int_{-r}^{r} \mathrm{d}q_1 \int_{-\sqrt{r^2-q_1^2}}^{\sqrt{r^2-q_1^2}} \mathrm{d}q_2 \int_{-\sqrt{r^2-q_1^2-q_2^2}}^{\sqrt{r^2-q_1^2-q_2^2}} \mathrm{d}q_3 \, \frac{2r}{\sqrt{r^2 - q_1^2 - q_2^2 - q_3^2}}.
\end{aligned}$$

[3] From the microcanonical distribution of question [1] with $N = 2$, derive a distribution function $P(v_1)$ for the absolute value of the velocity of particle 1, v_1, that is, $v_1 = |\boldsymbol{v}_1|$; $P(v_1)\mathrm{d}v_1$ gives the probability that v_1 is in the interval between v_1 and $v_1 + \mathrm{d}v_1$.

In the microcanonical distribution, any state with v_1 larger than R from question [1] does not appear. Therefore $P(v_1) = 0$ for $v_1 > R$.

For $v_1 < R$, $P(v_1)\mathrm{d}v_1$ is proportional to the size of the microcanonical distribution between v_1 and $v_1 + \mathrm{d}v_1$.

Derive $P(v_1)$ from the microcanonical distribution function. It is not necessary to normalize $P(v_1)$.

Next, consider the general case with N particles in the system. The surface size $S_d(r)$ of a d dimensional sphere with radius r is expressed by the following integration with $(d-1)$ variables, q_1, q_2, $\cdots$, and q_{d-1}:

$$S_d(r) = \int_{-r}^{r} \mathrm{d}q_1 \int_{-\sqrt{r^2-q_1^2}}^{\sqrt{r^2-q_1^2}} \mathrm{d}q_2 \int_{-\sqrt{r^2-q_1^2-q_2^2}}^{\sqrt{r^2-q_1^2-q_2^2}} \mathrm{d}q_3 \cdots \int_{-\sqrt{r^2-q_1^2-q_2^2-\cdots-q_{d-2}^2}}^{\sqrt{r^2-q_1^2-q_2^2-\cdots-q_{d-2}^2}} \mathrm{d}q_{d-1} \, \frac{2r}{\sqrt{r^2 - q_1^2 - q_2^2 - \cdots - q_{d-1}^2}}.$$

[4] Let the total kinetic energy E be $N\varepsilon$, where $\varepsilon > 0$ denotes the averaged kinetic energy per particle. Derive a distribution function $P(v_1)$ of the absolute value of the velocity of the particle 1, that is, $v_1 = |\boldsymbol{v}_1|$, from the microcanonical distribution function. It is not necessary to normalize $P(v_1)$.

[5] Show that the distribution function obtained in question [4] reproduces the Maxwell velocity distribution in the limit of $N \to \infty$.

The following formula may be used without proof:

$$\lim_{N\to\infty} \left(1 + \frac{a}{N}\right)^N = \mathrm{e}^a.$$

[6] From the answer to question [5], derive the temperature T of this system as a function of ε. The Boltzmann constant k_B may be used.

Problem 3

A small sphere of radius a with mass m_0 is charged homogeneously with the surface electric charge density σ. The sphere moves toward the positive direction along the x-axis with a constant velocity v ($v \ll c$, c being the speed of light) in the vacuum of free space and in the absence of an external electric field. In the following, assume that the electric field moves together with the sphere at the velocity v keeping a spherical symmetric form as in the $v = 0$ case. Here, the self-energy associated with the charge can be neglected.

[1] The center of the sphere passes the origin O at a time t. Express the electric flux density $D(\boldsymbol{r}, t) = \mid \boldsymbol{D}(\boldsymbol{r}, t) \mid$ at a point $\boldsymbol{r}$ at this time t in terms of a, σ, and $r = \mid \boldsymbol{r} \mid$, where $a < r$.

[2] Let the magnetic field be $\boldsymbol{H}(\boldsymbol{r}, t)$. Determine the relation between $\boldsymbol{D}$ and $\boldsymbol{H}$.

[3] Consider a circle C passing through a position $\boldsymbol{r}$ with its center located on the x-axis as shown in the figure. The circle C is set perpendicular to the x-axis. A magnetic field $\boldsymbol{H}$ due to the moving sphere points along the circle C. Integrate the magnetic field $\boldsymbol{H}$ along the circle C, and express the result in the form of an integral of the electric flux density $\boldsymbol{D}$.

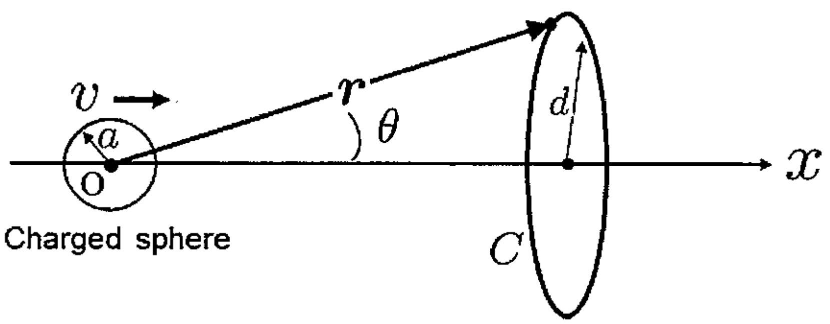
[4] Consider that an infinitesimal time Δt has elapsed from the time t. Express the change $\Delta\Psi$ of the electric flux, which penetrates the circle C, in terms of a, σ , r, v , θ, and Δt. Here, θ is the angle between the x-axis and $\boldsymbol{r}$.

[5] Using the answers to questions [3] and [4] above, express $H = \mid \boldsymbol{H}(\boldsymbol{r}, t) \mid$ in term of a, v, r, θ, and σ.

[6] Let the magnetic energy density induced by the magnetic field H be $u(r, t)$. Express $u(\boldsymbol{r}, t)$ in terms of H taking the permeability of vacuum to be μ_0.

[7] Determine the total magnetic energy associated with the motion of the sphere by integrating $u(\boldsymbol{r}, t)$ obtained in question [6] over all space outside of the sphere.

[8] Determine the energy (the kinetic energy plus the magnetic energy) associated with the motion of this small sphere, using the answer in question [7]. From this result, the charged sphere is regarded as possessing a heavier mass than the mass m_0. Describe its physical meaning in 30~60 words.



Problem 4

In the following, consider the phonon specific heat at a constant volume, C_V, for (a) a three-dimensional cubic lattice structure and (b) a two-dimensional square lattice structure, respectively.

The phonon dispersion relations are linear in the low frequency region. That is, for sufficiently small ω,

(a) $\omega = vk_{xyz} \quad (k_{xyz} = \sqrt{k_x^2 + k_y^2 + k_z^2})$,

(b) $\omega = vk_{xy} \quad (k_{xy} = \sqrt{k_x^2 + k_y^2})$,

where ω is the angular frequency, k_x, k_y, k_z are the x-, y-, and z-components of the wave vector $\boldsymbol{k}$, and v is the velocity of sound.

[1] The phonon density of states is denoted by $D(\omega)$, where the number of phonon states having angular frequency between ω and $\omega + \mathrm{d}\omega$ is given by $D(\omega)\mathrm{d}\omega$. This quantity is proportional to ω^p in the low ω region where the phonon dispersion relation is linear as described above. Determine the value of p for (a) the three-dimensional cubic lattice system and (b) the two-dimensional square lattice system. Describe the details of the calculation.

[2] In the low temperature limit, C_V is proportional to T^q, where T is the absolute temperature. Determine the value of q for (a) the three dimensional cubic lattice system and (b) the two dimensional square lattice system. Describe the details of the calculation.

Next, let us consider phonon specific heat at constant volume C_V for a layered system having strong in-plane bonding and weak out-of-plane bonding (see Fig. 2). Figure 3 shows the phonon dispersion relations in the in-plane direction and out-of-plane direction. The dispersion relation is significantly different between the two directions, and $\omega_a \ll \omega_b$. Although in principle there are three phonon modes in total, we assume for simplicity that these three modes are degenerate.

[3] Assign which of Fig. 3 (i) and (ii) is the in-plane or out-of-plane dispersion relation, and explain the reason based on the expected elastic properties of the system.

[4] The iso-energy surface of phonons satisfies $\omega(\boldsymbol{k}) = \omega_0$ in momentum space, where $\omega(\boldsymbol{k})$ is the phonon dispersion relation. For the phonons whose dispersion relations are shown in Fig. 3, explain the shape of the iso-energy surface for $\omega_0 \ll \omega_a$ and $\omega_a \ll \omega_0 \ll \omega_b$.

[5] For the phonon with the dispersion relation shown in Fig. 3, $D(\omega)$ is approximately proportional to ω^r for the two regimes $\omega \ll \omega_a$ and $\omega_a \ll \omega \ll \omega_b$. Determine the value of r for the respective regimes and explain the reasoning in detail.

[6] C_V is approximately proportional to T^s for the three regimes $k_\mathrm{B}T \ll \hbar\omega_a$, $\hbar\omega_a \ll k_\mathrm{B}T \ll \hbar\omega_b$, and $\hbar\omega_b \ll k_\mathrm{B}T$. Determine the value of s for the respective regimes and explain the reasoning in detail. Here, k_B and $\hbar$ are the Boltzmann constant and the Planck constant divided by 2π, respectively.

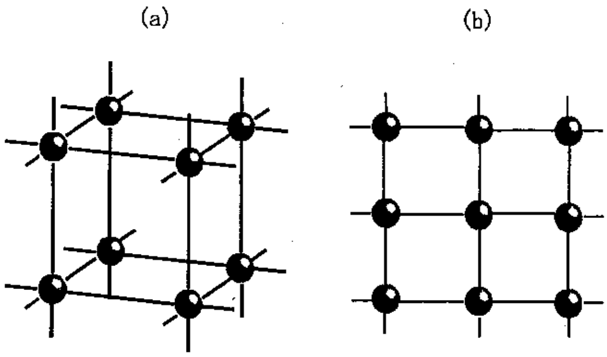


Figure 1: (a) Three dimensional cubic lattice structure. (b) Two dimensional square lattice structure.

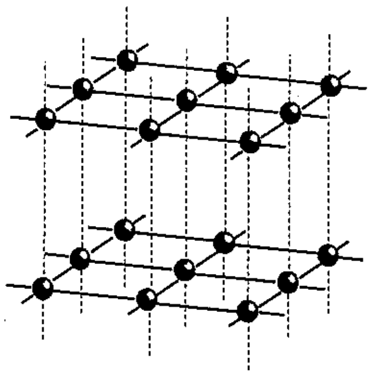


Figure 2: Layered system

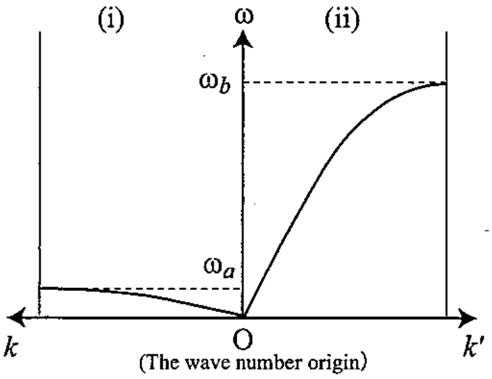


Figure 3: Phonon dispersion relation of the layered system in the first Brillouin zone. The horizontal axes k in (i) and k' in (ii) show the wave number for either the in-plane direction or out-of-plane direction. The size of the in-plane dispersion relation is significantly different from that of the out-of-plane dispersion relation, $\omega_a \ll \omega_b$.