Department of Applied Physics
Entrance Examination Booklet

Physics I

(Answer the 2 Problems in this Booklet)

August 28 (Tuesday) 9:30 – 11:30, 2012

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the two problems in this booklet.
4. Use one answer sheet for each problem (two answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Three solid cylinders A, B, and C of radius a, length l, and mass m, with rough surface are on a horizontal floor with homogeneous surface roughness. As shown in Fig. 1, cylinders A and B are placed on the floor aligned laterally in contact with each other, and cylinder C is placed stably on the two cylinders A and B aligned laterally so that the central axes of the three cylinders form an equilateral triangular prism. The amplitude of the gravitational acceleration is given by g. Answer the following questions, assuming that all the three cylinders are rigid bodies with a homogeneous density, and that no forces are exerted between cylinders A and B.

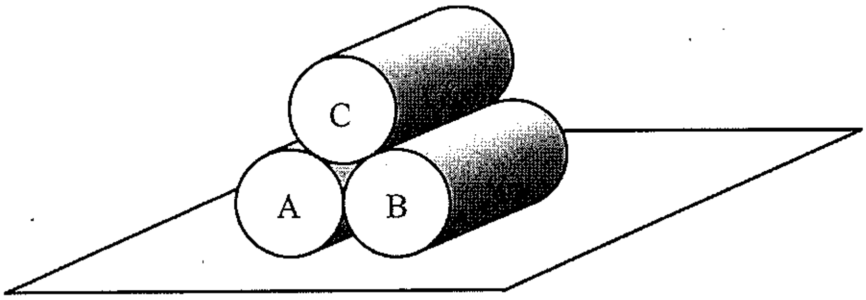


Fig. 1

[1] Supposing $\boldsymbol{F}$ and $\boldsymbol{F}'$ are frictional forces exerted on cylinder B from cylinder C and from the floor, respectively, draw the directions of $\boldsymbol{F}$ and $\boldsymbol{F}'$ as well as the points at which $\boldsymbol{F}$ and $\boldsymbol{F}'$ act. Find a formula relating the magnitudes of the two frictional forces $\boldsymbol{F}$ and $\boldsymbol{F}'$.

[2] In order for this arrangement of cylinder C on the two cylinders A and B to be stable, the static friction coefficient μ between the cylinders and the static friction coefficient μ' between each cylinder and the floor must satisfy certain conditions. Find these conditions.

Consider only one cylinder A described above moving on the same floor. As shown in Fig. 2, at time $t = 0$, the center of mass of the cylinder is at position S, the velocity of the center of mass is v_0 along the right direction, and the counterclockwise angular velocity around the central axis is ω_0. The cylinder initially moves to the right with the rotation slipping on the floor. Then, at position T, the cylinder changes its direction of movement to the left, and before coming back to the position S begins to roll on the floor without slipping. Let μ'' the dynamical friction coefficient between the cylinder and the floor. Answer the following questions. Note that rolling friction between the cylinder and the floor is negligible.

[3] Show the moment of inertia of the cylinder around the central axis is given by $\frac{1}{2}ma^2$.

[4] Find the distance between the two positions S and T. Find the angular velocity of the cylinder at the position T.

[5] Find the velocity of the cylinder when it begins to roll without slipping on the floor.

[6] Find the conditions of ω_0 for the cylinder to come back to the position S while having achieved rolling without slipping.

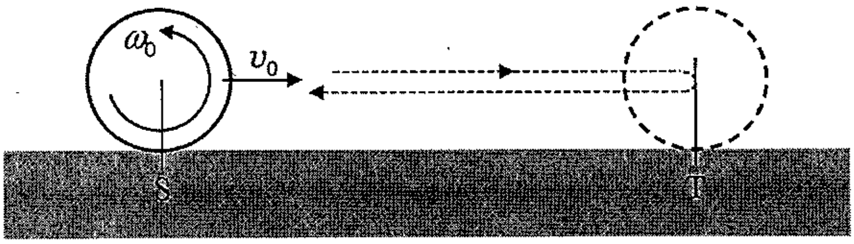
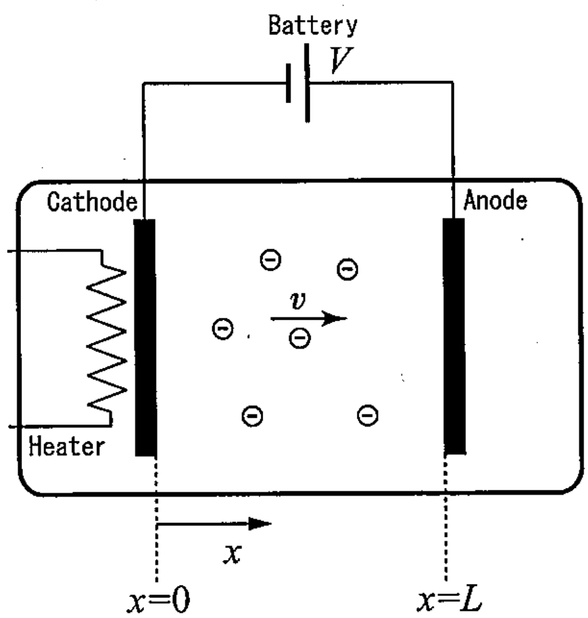


Fig. 2

Problem 2

As shown in the figure below, two planar electrodes are set parallel in a vacuum chamber with a separation of L; a battery of voltage V is connected to the electrodes. When the negative electrode (cathode) is heated with a heater, thermionic electron emission occurs and emitted electrons travel from the cathode to the positive electrode (anode); an electric current begins to flow. As the temperature of the cathode is raised, the emission rate of the thermionic electrons is increased. As a result, the electric current becomes large. However, it is known that there is a temperature range where the current is almost constant while the cathode temperature is changed. This phenomenon is caused by the space charge distribution originating from the emitted electrons that suppresses the emission of the electrons from the cathode. Here, we consider this situation (current flow independent of cathode temperature) with the following model.

We set the x axis perpendicular to the electrode plane and its origin ($x = 0$) at the position of the cathode. Since the size of the electrodes is large, we assume that the motion of the electrons is only along the x axis and the electric field, charge distribution and current distribution also depend only on x. Here, the force acting on an electron is assumed to originate from the mean electric field generated by the electrostatic potential $\phi(x)$ which is determined by the electron number density $n(x)$.



The current density $i(x)$ is given by

$$i(x) = -en(x)v(x), \tag{1}$$

where e is the elementary charge ($e > 0$) and $v(x)$ is the velocity of the electron at the position x, respectively.

The electrostatic potential $\phi(x)$ satisfies the following equation as

$$\varepsilon_0 \frac{d^2\phi(x)}{dx^2} = en(x), \tag{2}$$

where ε_0 is the permittivity of vacuum.

Since the initial velocity of the electron at the cathode is quite small we take it to be negligible; we assume that $v(0) = 0$. We also set $\phi(0) = 0$ and assume that the magnitude of the electric field at $x = 0$ is 0. The electron mass is denoted by m.

Answer the following questions.

[1] Express $v(x)$ using $\phi(x)$, e and m.

[2] In a stationary state, the current density $i(x)$ is a constant, $-i_0$ ($i_0 > 0$), independent of x where $0 < x < L$. Explain the reason.

[3] From [1] and [2], eq.(2) can be written as

$$\frac{d^2\phi(x)}{dx^2} = A\phi(x)^{\alpha}. \tag{3}$$

Find the expression for A using m, e, i_0 and ε_0. Obtain the value of α.

[4] Solve the equation obtained in [3] and express $\phi(x)$ using m, e, i_0, ε_0 and x.

[5] Find the relation between i_0 and V.

Department of Applied Physics
Entrance Examination Booklet

Physics II

(Answer 3 Problems among the 4 Problems in this Booklet)

August 28 (Tuesday) 13:00 – 16:00, 2012

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
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Examinee number	No.

Write down your examinee number above

Problem 1

The Hamiltonian for a charged particle confined in a one-dimensional harmonic potential is given as follows:

$$\hat{H}_0 = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{X}^2.$$

Here, m is the mass of the particle and ω_0 is the angular frequency of the harmonic oscillator. The momentum and position operators of the particle are denoted by $\hat{P}$ and $\hat{X}$, and they satisfy a commutation relation $[\hat{X}, \hat{P}] = i\hbar$. Here, $\hbar$ is the Planck constant divided by 2π. Creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators are defined as follows:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega_0}\hat{X} - \frac{i}{\sqrt{m\omega_0}}\hat{P}\right), \quad \hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega_0}\hat{X} + \frac{i}{\sqrt{m\omega_0}}\hat{P}\right).$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ satisfy a commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. One can rewrite the Hamiltonian $\hat{H}_0$ in the following form:

$$\hat{H}_0 = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega_0\left(\hat{N} + \frac{1}{2}\right).$$

The operator $\hat{N}$ is defined as $\hat{N} = \hat{a}^\dagger\hat{a}$, and its eigenvalue n takes non-negative integer values. We use $|n\rangle$ for eigenstates of $\hat{N}$, as in $\hat{N}|n\rangle = n|n\rangle$. The creation and annihilation operators satisfy the following relations: $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. The particle does not have internal degrees of freedom.

Consider the behavior of the system under the perturbations given below.

[1] Suppose that a transition between eigenstates $|n\rangle$ and $|l\rangle$ is induced by a perturbative Hamiltonian $\hat{H}' = -\hat{\mu}E(t)$. An electric dipole transition is allowed if $\langle l|\hat{\mu}|n\rangle \neq 0$ when we take only the lowest order of perturbation. Here, $\hat{\mu} = q\hat{X}$ is the electric dipole operator, $E(t)$ is the electric field that depends on time, and q is the charge of the particle. Show the levels to which the electric dipole transition from $|n\rangle$ is allowed.

[2] Suppose an anharmonic perturbation $\hat{H}_{\mathrm{AH}} = \lambda\hat{X}^4$ ($\lambda > 0$) is added to the original Hamiltonian $\hat{H}_0$. Calculate the energy shift of eigenstate $|n\rangle$ up to the first order in λ.

[3] The anharmonic perturbation $\hat{H}_{\mathrm{AH}}$ introduced in [2] can affect electric dipole transitions. Find the levels to which electric dipole transitions from $|n = 8\rangle'$ are allowed. Here, $|n\rangle'$ denotes a new eigenstate of the Hamiltonian including the anharmonic perturbation. For finding the allowed transitions, consider the transition matrix element $\langle l|'\hat{\mu}|n\rangle'$ up to the first order in λ.

Next, consider the dynamics of the system with the original Hamiltonian $\hat{H}_0$ (without any perturbations).

[4] Define $\hat{X}(t)$ as the position operator in the Heisenberg representation, whose time evolution is governed by the Hamiltonian $\hat{H}_0$. Express $\hat{X}(t)$ as a linear combination of $\hat{X}(0)$ and $\hat{P}(0)$.

[5] In classical mechanics, the position of a particle in a harmonic potential oscillates with the angular frequency ω_0. From the time evolution of the position operator $\hat{X}(t)$ obtained in [4], find the expectation value $\langle n|\hat{X}(t)|n\rangle$ of the position for an eigenstate $|n\rangle$. Further, show an example of a state whose expectation value of position oscillates with a finite amplitude.

Problem 2

Liquid ^{4}He is known to become a superfluid below 2.2 K under ambient pressure. To discuss this phenomenon, let us consider a simple model of a three-dimensional ideal Bose gas in the thermal equilibrium. The volume of the gas $V = L^3$ is sufficiently large so that it can be regarded to be in the thermodynamic limit. The mass of the boson is m, the momentum is p, the energy is $E_p = p^2/2m$, and the spin is 0. Here k_B is the Boltzmann constant and $\hbar$ is the Planck constant divided by 2π.

If necessary, you may use the following integral formulae,

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = \Gamma(3/2)\zeta(3/2), \quad \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx = \Gamma(5/2)\zeta(5/2),$$

where values are given by

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, \quad \zeta(3/2) \fallingdotseq 2.61, \quad \zeta(5/2) \fallingdotseq 1.34.$$

[1] Using the temperature T and the chemical potential μ, write down the Bose-Einstein distribution function $f_{\rm BE}(E)$ which represents the expected number of particles in a single particle state with the energy E. Specify the physically allowed range of μ.

[2] If the energy E is fixed, how does $f_{\rm BE}(E)$ depend on μ? Draw a schematic figure of $f_{\rm BE}(E)$ as a function of μ.

[3] Derive the density of states $D(E)$ of the system as a function of E, m, and V.

[4] When the system obeys the Bose-Einstein distribution function in [1], the particle number $N_{\rm t}$ can be computed from the following equation:

$$N_{\rm t} = \int_0^\infty D(E) f_{\rm BE}(E) dE.$$

Express the upper limit of $N_{\rm t}$, $N_{\rm t}^{\rm max}$, using m, V, and T.

Describe how $N_{\rm t}^{\rm max}$ behaves when $T \to 0$.

[5] When the temperature is decreased with the total particle number being fixed at N, $N_{\rm t}^{\rm max}$ becomes smaller than N below a certain temperature $T_{\rm c}$. Find the temperature $T_{\rm c}$.

[6] Describe the distribution of the Bose particles as a function of energy at temperature T below $T_{\rm c}$.

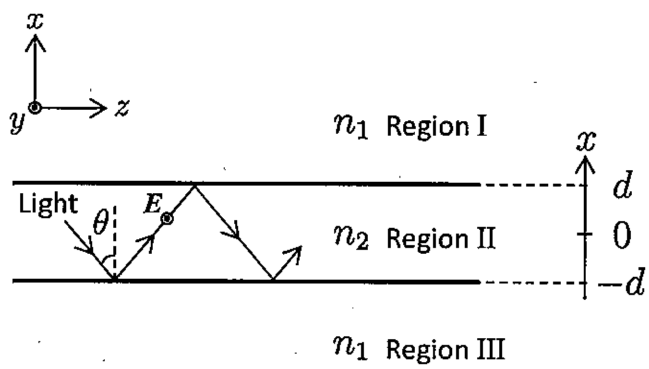
[7] Find the total energy U and the heat capacity at constant volume, C_V, at temperature T below $T_{\rm c}$ in terms of m, V, and T.

[8] Calculate $T_{\rm c}$ to one significant digit for the case of ^{4}He using the result of [5]. If necessary, you may use the following values; the density of liquid ^{4}He: $1.5 \times 10^2\,{\rm kg/m^3}$, atomic mass of ^{4}He: $6.7 \times 10^{-27}\,{\rm kg}$, $\hbar = 1.1 \times 10^{-34}\,{\rm J \cdot s}$, and $k_B = 1.4 \times 10^{-23}\,{\rm J/K}$.

[9] What is (are) the possible origin(s) of the difference between $T_{\rm c}$ obtained in [8] and the experimental value 2.2 K of the superfluid transition temperature in ^{4}He, that is (are) neglected in the above simple model?

Problem 3

Consider the propagation of light in a planar waveguide as shown in the figure below. Region II is a dielectric thin film whose thickness and refractive index are $2d$ and n_2, respectively. Regions I and III are thick dielectrics, and their refractive indices are the same, n_1 $(1 < n_1 < n_2)$. Light propagates along the $+z$ direction. The dielectric thin film and the thick dielectrics are large in the y and z directions. Assume that the electromagnetic field does not depend on y. Consider the situation where the electric field vector is parallel to the y axis. The wavenumber in vacuum is $k_0 = \omega/c$, where ω is the angular frequency of light and c is the speed of light.



[1] At first, let's examine the propagation of light in terms of the geometrical optics. As shown in the figure above, a ray is reflected at the boundaries ($x = \pm d$), and proceeds toward the $+z$ direction. The incident angle is θ $(0 < \theta < \pi/2)$. Let β be the z component of the wavenumber from the viewpoint of wave optics. Using the θ from geometrical optics, β can be written as $\beta = k_0 n_2 \sin\theta$. Light can be confined and propagate in Region II for a certain range of β. Express this range of β by using k_0, n_1 and n_2.

[2] Next, let's describe the propagation of light by Maxwell's equations. Consider that the wavenumber of the electromagnetic field propagating toward the $+z$ direction is given by β (> 0). The electric field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$ can be written as,

$$\begin{aligned} \boldsymbol{E} &= (0, E_y(x), 0)e^{i(\omega t - \beta z)}, \\ \boldsymbol{H} &= (H_x(x), H_y(x), H_z(x))e^{i(\omega t - \beta z)}. \end{aligned}$$

Here we note that the amplitude of the electromagnetic field does not depend on y, and the electric field vector is parallel to the y axis. Derive the equations for $E_y(x)$, $H_x(x)$, $H_y(x)$ and $H_z(x)$ from Maxwell's equations $\boldsymbol{\nabla} \times \boldsymbol{H} = \epsilon_j \frac{\partial \boldsymbol{E}}{\partial t}$ and $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$. Here ϵ_j is permittivity ($j = 1$ for Regions I and III, $j = 2$ for Region II), and μ_0 is the permeability of vacuum. Assume that permeability is μ_0 in the whole Regions I, II and III. Note that there is neither true charge nor conduction current.

[3] As expected from geometrical optics, the magnetic field vector may have a finite z component when the electromagnetic field propagates in Region II. Using the equations derived in [2], show $H_z(x) \neq 0$ if the electromagnetic field exists. Note that $\beta \neq k_0 n_2$ and $\mu_0 \epsilon_j = n_j^2/c^2$ $(j = 1, 2)$.

The condition for β derived in [1] is only a necessary condition to confine light in Region II. In reality, β takes discrete values. In what follows, let's examine this.

[4] Show that the following differential equation for $E_y(x)$ is obtained from the equations derived in [2]:

$$\frac{d^2E_y(x)}{dx^2} + (k_0^2 n_j^2 - \beta^2)E_y(x) = 0, \quad j = \begin{cases} 1 & (|x| > d) \\ 2 & (|x| < d) \end{cases} .$$

[5] From the differential equation for $E_y(x)$ and the boundary conditions (i.e., the continuity of $E_y(x)$ and dE_y/dx at the boundaries), the eigenfunction $E_y(x)$ and eigenvalue β can be obtained. For simplicity, assume that $E_y(x)$ is an even function, i.e., $E_y(x) = E_y(-x)$. When the eigenvalue equation is written in the form of

$$\tan(A(\beta)) = B(\beta),$$

find $A(\beta)$ and $B(\beta)$. Here assume that β satisfies the condition derived in [1].

[6] Taking into account $\tan(A(\beta)) = \tan(A(\beta) - m\pi)$ (m is integer), the eigenvalue equation of [5] is rewritten as,

$$F(\beta) \equiv A(\beta) - \arctan(B(\beta)) = m\pi,$$

where we set $\arctan(0) = 0$. By calculating the range of $F(\beta)$, show that there is at least one β which satisfies the eigenvalue equation. In addition, find the condition that there is only one β which satisfies the equation, and draw a rough sketch of $E_y(x)$ in that condition.

Problem 4

Consider a one-dimensional lattice consisting of a unit cell containing two atoms with mass m. As shown below, two kinds of interactions described by the spring constants C_1 and C_2 ($C_1 > C_2 > 0$) are acting between the neighboring atoms alternatively. The length of the unit cell in the equilibrium state is a. When the atoms are oscillating slightly along the one-dimensional direction, there exist two kinds of specific oscillation modes (optical and acoustic modes). Answer the following questions. Here we take the right direction positive.

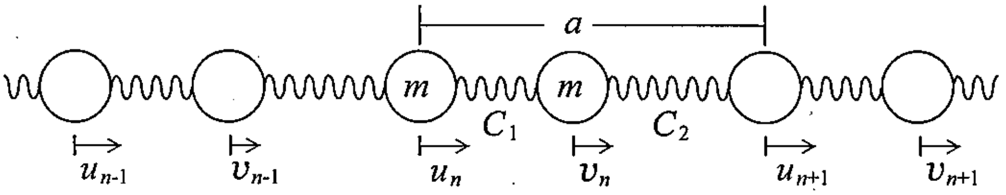


Fig. 1

[1] Write down Newton's equations of motion for the atoms in the nth unit cell using u_n and v_n, which represent small displacements from the equilibrium positions, and derive relations between the angular frequency ω and the wave number k for the two types of modes (dispersion relations of phonons).

[2] Illustrate the two kinds of dispersion relations obtained in [1] for the 1st Brillouin zone. Find the values of ω at $k = 0$ and π/a, and the group velocities in the long-wavelength limit.

[3] Explain briefly the manner of oscillatory motion for the atoms in the two kinds of phonon modes in the long-wavelength limit.

Next, consider inelastic scattering experiments of photons and neutrons to study the phonon dispersion relation of a model crystal shown in Fig. 2. The model crystal consists of two atoms with mass m, and the lattice constant along the x axis is a. Here, we assume that the atomic displacement is confined along the x direction, and consider the wave vector of phonon only for $\boldsymbol{q} = (q, 0, 0)$. In addition, we assume that the dispersion relation along the x direction can be approximated by the above one-dimensional lattice. Here, the wave vector $(\pm\pi/a, 0, 0)$ corresponds to the 1st Brillouin zone boundary. Answer the following questions.

[4] The scattering of photons from visible laser light by the acoustic phonons is called Brillouin scattering. Here, the angular frequencies of incident light, scattered light, and the created phonon are ω_i, ω_f, and ω_B, and their wave vectors inside the sample are $\boldsymbol{K}_\mathrm{i}$, $\boldsymbol{K}_\mathrm{f}$, and $\boldsymbol{q}_\mathrm{B}$, respectively. Then the following relation, $\omega_\mathrm{i} - \omega_\mathrm{f} = \omega_\mathrm{B}$, is known to be satisfied. Given the scattering angle of laser light inside the sample θ, draw a diagram to illustrate the relation between $\boldsymbol{K}_\mathrm{i}$, $\boldsymbol{K}_\mathrm{f}$, and $\boldsymbol{q}_\mathrm{B}$. In addition, write down the relation between $|\boldsymbol{q}_\mathrm{B}|$ and $|\boldsymbol{K}_\mathrm{i}|$ assuming $\omega_\mathrm{i} \gg \omega_\mathrm{B}$.

[5] Given the refractive index n of the model crystal, the velocity of light c in the vacuum, and the sound velocity v_a (group velocity of the acoustic mode in the long-wavelength limit), write down the relation between the angular frequency ω_B and ω_i. Here, consider the long-wavelength limit of phonons.

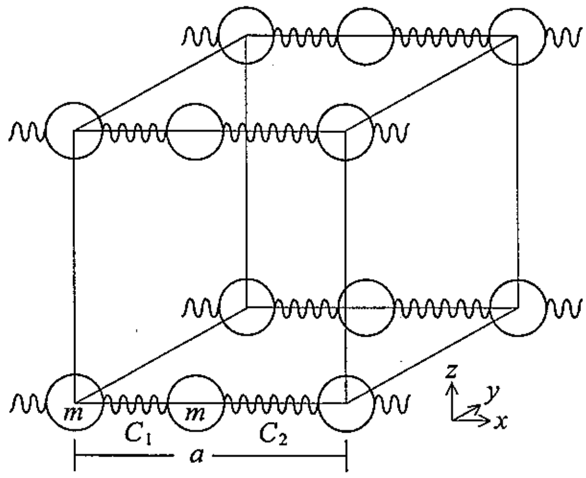


Fig. 2

[6] Using laser light with a wavelength of 680 nm directed along the x axis as an incident beam and fixing the scattering angle θ at 180°, the energy spectrum of the scattered light is measured. The observed energy shift $\hbar\omega_B$, if expressed as $\hbar\omega_B = hc/\lambda_B$ ($\hbar$ is the Planck constant h divided by 2π), is equivalent to $1/\lambda_B = 3.0\ \mathrm{cm}^{-1}$. Calculate the sound velocity v_a to two significant digits, using the velocity of light, 3.0×10^8 m/s, and the refractive index, 2.4.

[7] Consider inelastic scattering of neutrons by the phonons. With the wave vector of phonons and the wavelength of scattered neutrons fixed at $\boldsymbol{q} = (\pi/a, 0, 0)$ and λ_f, respectively, inelastic phonon scattering is observed when the wavelengths of incident neutrons are λ_{i1} and λ_{i2} ($\lambda_{i1} < \lambda_{i2}$). Express C_1 and C_2 in terms of m, λ_f, λ_{i1}, λ_{i2}, and the mass of neutron M_n.

[8] Based on the above results, express the lattice parameter a of the model crystal in terms of v_a, λ_f, λ_{i1}, λ_{i2}, and M_n.