Department of Applied Physics
Entrance Examination Booklet

Physics I

(Answer the 2 Problems in this Booklet)

August 30 (Tuesday) 9:30 – 11:30, 2011

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the two problems in this booklet.
4. Use one answer sheet for each problem (two answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

A particle of mass m is connected by a massless fine string of length ℓ_0 to a point A on the edge of a fixed disc with radius a, as shown in Fig.1. Given an initial speed $v_0 > 0$ in the negative x-direction at time $t = 0$, let us study the winding motion of the particle around the disc (see Fig.2). This motion takes place in the xy-plane. To describe the motion, let $\varphi(t)$ be the angle between the x-axis and the radius vector $\overrightarrow{\mathrm{OQ}}$, where the string separates from the disc at Q . Only a tensile force $\boldsymbol{T}(t)$ from the string acts upon the particle, and the string remains taut during the motion. Neglect gravitational force and any frictional forces.

[1] In this motion of the particle, (i) the kinetic energy is conserved, while (ii) the angular momentum around the origin O is not conserved. Explain the physical reasons briefly for (i) and (ii), respectively.

[2] Derive the expressions for each of the following quantities of the particle in terms of $\varphi(t)$ and $\frac{d}{dt}\varphi(t)$: the position vector $\boldsymbol{r}(t) = (x(t), y(t))$, the velocity vector $\boldsymbol{v}(t) = \frac{d}{dt}\boldsymbol{r}(t) = (\frac{d}{dt}x(t), \frac{d}{dt}y(t))$, and the angular momentum around O, $L(t) = m(x(t)\frac{d}{dt}y(t) - y(t)\frac{d}{dt}x(t))$.

[3] According to the considerations in question [1], the magnitude of the velocity vector satisfies $|\boldsymbol{v}(t)| = v_0$ (= const.). Solving this differential equation, find $\varphi(t)$ as a function of t. And find the time τ when the whole string is wound up around the disc (in other words, the time when the particle hits the disc).

[4] Find the tensile force vector $\boldsymbol{T}(t) = (T_x(t), T_y(t))$. The final result should be expressed in terms of $\varphi(t)$ without $\frac{d}{dt}\varphi(t)$ and $\frac{d^2}{dt^2}\varphi(t)$. Based on the expression of $\boldsymbol{T}(t)$, discuss the physical picture of this motion during an infinitesimal time interval.

[5] Find the moment of the tensile force around O, $N(t) = x(t)T_y(t) - y(t)T_x(t)$. Show that the relation $\frac{d}{dt}L(t) = N(t)$ holds.

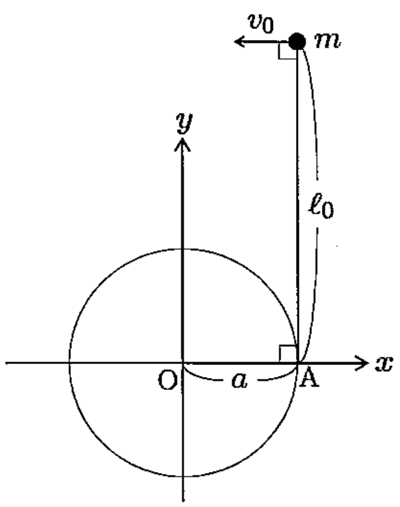


Figure 1: $t = 0$

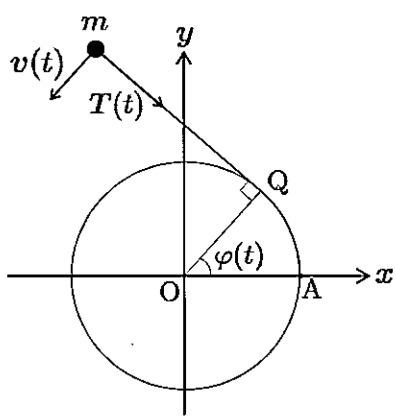


Figure 2: $t > 0$

Problem 2

Consider Joule heating in a cylindrical conductor as shown in Figure. The radius of the cylindrical conductor is a, and its length is L. The effects of both ends are negligible assuming L is much larger than a. The SI unit system should be used for expressions.

[1] Consider the following model for the motion of each individual electron.

A static electric field is applied along the axial direction of the cylindrical conductor. Here, we only consider the electron motion parallel to the electric field. The electron is constantly accelerated by the electric field, and is then scattered at the time intervals of 2τ. At the scattering, velocity of the electron becomes zero and kinetic energy of the electron is totally lost. After the scattering, the electron repeats this motion.

Answer the following questions by using the quantities provided below. Let E be strength of an electric field, m the electron mass, q the electron charge, and n the electron density of the conductor.

[1.1] Derive an expression for the average velocity v_a.

[1.2] Show that the induced current is proportional to the voltage applied to the conductor (Ohm's law). In addition, write down the electric conductivity σ of the conductor.

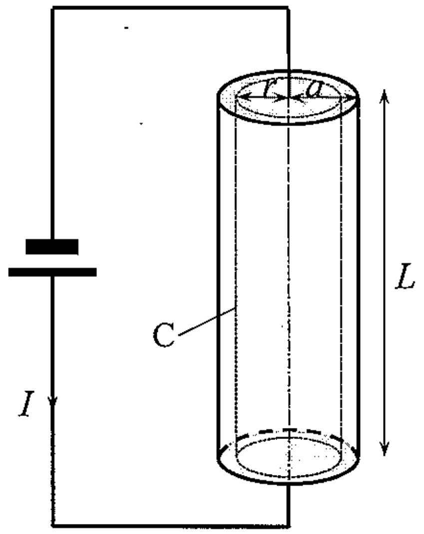
[1.3] Derive an expression for the Joule heat in the conductor per unit volume and unit time, J, in terms of the conductivity σ, the current I, and the radius a. In addition, show that J is equal to the loss of the kinetic energy of the electrons by the scatterings per unit volume and unit time.

[2] Consider Joule heating from the viewpoint of macroscopic energy flow. As shown in Figure, an external electric source is connected to the cylindrical conductor, and it supplies constant current I from the bottom face to the top face of the cylinder. Here, the electric current is small enough for the conductivity to be defined locally.

[2.1] The flux density of energy flow under the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ in a material is given by the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, where"$\times$" denotes vector product. In the case that the cylindrical conductor is uniform, find the direction and magnitude of $\boldsymbol{S}$ at a position with the distance r $(< a)$ from the cylindrical axis.

[2.2] As shown in Figure, consider a cylindrical volume region C with length L and radius r $(< a)$ of the conductor. Region C and the whole conductor have the common cylindrical axis. The energy flow into (or out from) the region C per unit time is obtained by integrating $|\boldsymbol{S}|$ on the side surface of C. Obtain the energy flow, and compare it with the Joule heat in region C. Based on these results, discuss about the conservation of energy in this system.

[2.3] Consider the case in which the cylindrical conductor is not uniform and its conductivity is a function of r alone, $\sigma(r)$. Express the current density $j(r)$ as a function of r. The answer may contain differential or integral symbols, if necessary. In the same way as the previous question [2.2], calculate the energy flow into (or out from) region C per unit time, and compare it with the Joule heat in region C.



Figure

Department of Applied Physics
Entrance Examination Booklet

Physics II

(Answer 3 Problems among the 4 Problems in this Booklet)

August 30 (Tuesday) 13:00 – 16:00, 2011

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the blank sheet of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider the spin operator $\boldsymbol{J} = (J_x, J_y, J_z)$ and the simultaneous eigenfunction $|J, m\rangle$ of $\boldsymbol{J}^2$ and J_z with the eigenvalues $J(J+1)\hbar^2$ and $m\hbar$, respectively, where J is an integer or a half-odd integer, and $m = J, J-1, ..., -J$. $J_\pm = J_x \pm iJ_y$ satisfy the following relations: $J_\pm |J, m\rangle = \hbar\sqrt{(J \mp m)(J \pm m + 1)}\,|J, m \pm 1\rangle$. $h = 2\pi\hbar$ is the Planck constant.

Below we consider a system of $J = 1$ in a static magnetic field along the z-axis, $\boldsymbol{B} = (0, 0, B)$ $(B > 0)$.

[1] Represent J_x, J_y, and J_z in the matrix form using the basis set of $\{|J, m\rangle\}$.

[2] Suppose that the Hamiltonian of the present system is given by $H_{\mathrm{M}} = -\gamma \boldsymbol{J} \cdot \boldsymbol{B}$ with a constant γ $(\gamma > 0)$. Find all the energy eigenvalues of the system.

[3] Suppose that an oscillating magnetic field $\boldsymbol{B}_{\mathrm{RF}} = (2B_{\mathrm{RF}} \cos\omega t, 0, 0)$ along the x-axis is applied to the present system $(B_{\mathrm{RF}} > 0)$. In order to induce transitions between the energy levels of H_{M} in question [2], we choose $\hbar\omega$ equal to the energy level spacing between the eigenstates of H_{M}. When $B_{\mathrm{RF}} \ll B$, the time dependence of $|\Psi(t)\rangle = \exp(-i\omega t J_z)|\varphi(t)\rangle$ is approximately described by $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H_{\mathrm{RF}}|\Psi(t)\rangle$. Here $|\varphi(t)\rangle$ is the quantum state of the system at a given time t and $H_{\mathrm{RF}} = -\gamma B_{\mathrm{RF}} J_x$. Find the time dependences of $|\Psi(t)\rangle$ and the expectation value of J_z by using the fact that $|\Psi(t)\rangle$ can be expressed as $|\Psi(t)\rangle = \sum_m c_m(t)|J, m\rangle$. Here we assume the initial state at $t = 0$ is the ground state of H_{M}.

[4] Suppose that the system discussed in question [2] is subject to the interaction $H_{\mathrm{Q}} = A\left(2J_x^2 - J_y^2 - J_z^2\right)$, and thus the total Hamiltonian of the system is $H = H_{\mathrm{M}} + H_{\mathrm{Q}}$. Here A $(A > 0)$ is a constant. Assume that H_{Q} is much smaller than H_{M}. Find all the energy eigenvalues of the system using the first-order perturbation theory, and show schematically the energy levels for both cases with and without H_{Q}.

Problem 2

Consider a gas made of indistinguishable N particles with mass m, which is confined in a one-dimensional box of length L. This one-dimensional gas is in a thermal equilibrium at an absolute temperature T. Here we assume that the temperature is high and the system obeys the Boltzmann statistics. You may assume that a cell of size given by the Planck constant h in one-particle phase space corresponds to a quantum state. Here the relation $N \gg 1$ is satisfied, and thus you may use Stirling's formula $\ln N! \simeq N \ln N - N$. You may ignore the spin degrees of freedom.

[1] First we consider gas A in which there are no interactions between particles. Determine the partition function for gas A, Z_{A}, and express it in terms of N, $\lambda = \sqrt{2\pi\hbar^2/mk_{\mathrm{B}}T}$, and L. Here $\hbar = h/2\pi$ and k_{B} is the Boltzmann constant. If necessary, you may use the relation $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$.

[2] Find the free energy of gas A, $F(T, L)$.

[3] Find the chemical potential of gas A, $\mu(T, L)$.

[4] Let us consider the validity of the Boltzmann statics at a high enough temperature. In the Boltzmann statistics, the expectation value of the particle number occupying energy level E_k is given by $\bar{n}_k = \exp((\mu - E_k)/k_{\mathrm{B}}T)$ and the relation $\bar{n}_k \ll 1$ is assumed. By using the result of question [3], express the condition for the above relation to be satisfied, in terms of L, N, and λ. Then, find the dimension of the physical quantity λ, describe its physical meaning, and explain the meaning of the condition you obtained.

[5] Next, consider gas B made of impenetrable particles, which interact via the following hard-core repulsive potential $v(R)$ (R: the center-to-center distance between particles):

$$v(R) = \begin{cases} \infty & \text{for } R < d \\ 0 & \text{for } R \geq d. \end{cases}$$

Both ends of the one-dimensional box of length L also interact with particles via hard-core potential. Namely, the distance of the center of a particle to an end cannot be less than $d/2$. Determine the partition function of gas B, Z_{B}. Here you may assume $L \gg Nd$. Find the equation of state for gas B and discuss the difference from that of gas A.

Problem 3

Consider the steady states of electromagnetic field oscillating in an angular frequency $\omega > 0$ when two media are separated by a planar interface. Neither true charges nor true currents exist at the interface and in the two media. In the following, bold letters represent vectors. Let ϵ_0 be the permittivity of vacuum, μ_0 the permeability of vacuum, and c the speed of light in vacuum.

Define an xyz coordinate system such that the planar interface is given by $z = 0$, and call the medium for $z < 0$ Medium 1 and the one for $z > 0$ Medium 2. Let ϵ_j, a real scalar, be the permittivity of Medium j ($j = 1, 2$) at the angular frequency ω. Both media have the permeability equal to μ_0. Here we only consider the states for which the electric field vector $\boldsymbol{E}(x, y, z, t)$ and the magnetic field vector $\boldsymbol{H}(x, y, z, t)$ at position (x, y, z) and at time t are written as

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}(x, z)e^{-i\omega t},$$
$$\boldsymbol{H}(x, y, z, t) = \boldsymbol{H}(x, z)e^{-i\omega t},$$

and we assume that the x- and z-components of vector $\boldsymbol{H}(x, z)$ are zero everywhere. When there are neither true charges nor true currents, the electromagnetic fields having time dependence in the form of $e^{-i\omega t}$ follow Maxwell's equations $\mathrm{div}\boldsymbol{D} = 0$, $\mathrm{div}\boldsymbol{B} = 0$, $\mathrm{rot}\boldsymbol{H} + i\omega\boldsymbol{D} = \boldsymbol{0}$, and $\mathrm{rot}\boldsymbol{E} - i\omega\boldsymbol{B} = \boldsymbol{0}$. The relations $\boldsymbol{D} = \epsilon_j\boldsymbol{E}$ and $\boldsymbol{H} = \mu_0^{-1}\boldsymbol{B}$ hold in Medium j ($j = 1, 2$). Answer the following questions.

[1] Consider the case where Medium 1 has a refractive index $n > 1$ and Medium 2 is vacuum (Fig. 1), implying that $\sqrt{\epsilon_1\mu_0} = n/c$ and $\sqrt{\epsilon_2\mu_0} = \sqrt{\epsilon_0\mu_0} = 1/c$. We want to determine the range of k_x for which the plane wave with wave-number vector $\boldsymbol{k}_{\mathrm{in}} = (k_x, 0, k_1)$ incident on the interface from the region of Medium 1 is totally reflected and the electric field in the region $z > 0$ takes a form of $\boldsymbol{E}(x, z) = \boldsymbol{E}_2 e^{ik_x x - \kappa_2 z}$. Here we assume that $k_x \geq 0$, $k_1 > 0$, and $\kappa_2 > 0$. First, using the formula $\mathrm{rot}(\mathrm{rot}\boldsymbol{E}) = \mathrm{grad}(\mathrm{div}\boldsymbol{E}) - \Delta\boldsymbol{E}$ and Maxwell's equations, derive the expressions for k_1 and κ_2 in terms of k_x, n, c, and ω. Note that Δ in the formula is Laplacian. Next, write down the range of k_x using ω, c, and n.

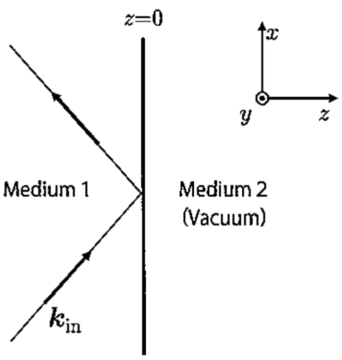


Figure 1

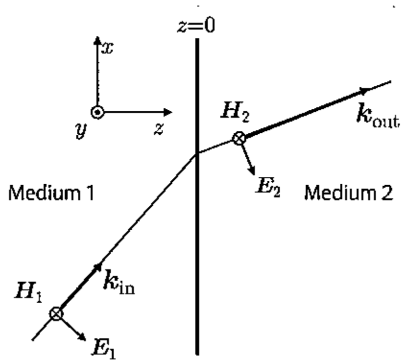


Figure 2

[2] Consider the case where $\epsilon_2 > \epsilon_1 > 0$ (Fig. 2). We want to determine the condition for k_x such that the plane wave with wave-number vector $\boldsymbol{k}_{\text{in}} = (k_x, 0, k_1)$ incident on the interface from the region of Medium 1 is perfectly transmitted through the interface with no reflection and propagates in Medium 2 as a refracted wave with wave-number vector $\boldsymbol{k}_{\text{out}} = (k_x, 0, k_2)$. Here we assume that $k_x \geq 0$ and $k_1 > 0$. In this case, the electromagnetic field is written as

$$\boldsymbol{E}(x,z) = \begin{cases} \boldsymbol{E}_1 e^{i(k_x x + k_1 z)} & (z < 0) \\ \boldsymbol{E}_2 e^{i(k_x x + k_2 z)} & (z > 0) \end{cases} \qquad \boldsymbol{H}(x,z) = \begin{cases} \boldsymbol{H}_1 e^{i(k_x x + k_1 z)} & (z < 0) \\ \boldsymbol{H}_2 e^{i(k_x x + k_2 z)} & (z > 0). \end{cases}$$

First, derive the expressions for $\zeta_1 \equiv E_{1,x}/H_{1,y}$ and $\zeta_2 \equiv E_{2,x}/H_{2,y}$ in terms of ω, k_1, k_2, ϵ_1, and ϵ_2. Here $E_{j,x}$ stands for the x-component of vector $\boldsymbol{E}_j$ and $H_{j,y}$ stands for the y-component of vector $\boldsymbol{H}_j$. Next, using the boundary conditions on the electromagnetic field at the interface, derive the expression for k_x in terms of ω, μ_0, ϵ_1, and ϵ_2.

[3] Suppose that Medium 1 is vacuum ($\epsilon_1 = \epsilon_0$) and Medium 2 is a metal satisfying $\epsilon_2 < -\epsilon_0$. In this case, consider a state with the form of

$$\boldsymbol{E}(x,z) = \begin{cases} \boldsymbol{E}_1 e^{ik_x x + \kappa_1 z} & (z < 0) \\ \boldsymbol{E}_2 e^{ik_x x - \kappa_2 z} & (z > 0) \end{cases} \qquad \boldsymbol{H}(x,z) = \begin{cases} \boldsymbol{H}_1 e^{ik_x x + \kappa_1 z} & (z < 0) \\ \boldsymbol{H}_2 e^{ik_x x - \kappa_2 z} & (z > 0) \end{cases}$$

with $\kappa_1 > 0$ and $\kappa_2 > 0$. From the relation between k_x and ω in order for such a state to exist, derive the expression for k_x in terms of ω, $|\epsilon_2|$, ϵ_0, and μ_0. Then, compare the magnitude of k_x to ω/c.

[4] In order to excite the state considered in question [3] by light, a configuration with a prism as in Fig. 3 is often used. Here the prism is placed very close to the surface of the metal, and the light is incident at an angle that would cause total reflection if the metal was absent. Based on the results obtained in questions [1] and [3], explain why this configuration is advantageous in comparison to the case where the light is directly shone on the surface of the metal.

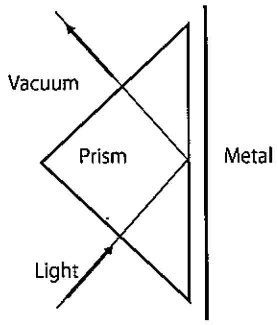


Figure 3

Problem 4

Consider how electronic states evolve when isolated atoms form a crystal using a one-dimensional model. Suppose that, in an isolated atom of element A, the outermost atomic orbital, $\phi_A(x)$, satisfies the following Schrödinger equation:

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + u(x))\phi_A(x) = E_A\phi_A(x),$$

where $h = 2\pi\hbar$ is the Planck constant, m is the electron mass, and $u(x)$ is the potential of A ion at the origin. When A atoms form a one-dimensional crystal with a periodicity of a as shown in Fig. 1, the Schrödinger equation is given by

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x))\psi(x) = E\psi(x), \tag{1}$$

where $U(x) = \sum_{j=1}^{M} u(x - ja)$ with M ($\gg 1$) being the number of atoms. Let us solve eq. (1) under the periodic boundary condition, $\psi(x) = \psi(x + Ma)$, provided that the atomic orbitals of A atom can be neglected except for ϕ_A. We ignore the electron-electron interaction.

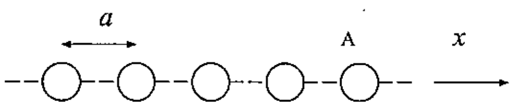


Figure 1: One-dimensional crystal composed of A atoms.

[1] Let us assume the solution for the present crystal as follows:

$$\psi_k(x) = \sum_{j=1}^{M} e^{ik\cdot(ja)}\phi_A(x - ja). \tag{2}$$

Find the condition that k should satisfy. You may ignore the normalization of wave function.

[2] Let $E(k)$ be the eigenenergy of $\psi_k(x)$. Find $E(k)$ (hint: plug $\psi_k(x)$ in eq. (1), then multiply both sides by $\phi_A^*(x)$, and integrate them over the whole space). Concerning the matrix elements of the overlap integrals and potential between atomic orbitals, only the following real-valued integrals should be taken into account in this procedure:

$$\int \phi_A^*(x - ma)\phi_A(x - na)dx = \delta_{mn},$$

$$\beta \equiv \int \phi_A^*(x)u(x \pm a)\phi_A(x)dx,$$

$$\gamma \equiv \int \phi_A^*(x)u(x)\phi_A(x \pm a)dx.$$

Here, δ_{mn} represents Kronecker's δ symbol.

[3] Plot $E(k)$ obtained in question [2] within the first Brillouin zone. Indicate E_A, β and γ in the plot and discuss briefly their signs and magnitudes in the light of the stability of atom and solid.

[4] Find the effective mass of the carrier in the present one-dimensional crystal with the energy dispersion $E(k)$ obtained in question [2] for the following two cases separately: an average electron number in ϕ_A per atom is (i) much less than one, and (ii) very close to, but slightly less than two.

Next consider a one-dimensional crystal composed of molecule AB formed by elements A and B as shown schematically in Fig. 2. Let $\phi_B(x)$ be the normalized outermost atomic orbital of B atom, E_B its energy, and $w(x)$ the potential of an isolated B ion. Concerning the matrix elements of the overlap integrals and potential between atomic orbitals, in addition to β and γ described in question [2], only the following two kinds of real-valued integrals between ϕ_A and ϕ_B within the same molecule should be taken into account:

$$\delta \equiv \int \phi_A^*(x)u(x)\phi_B(x)dx = \int \phi_A^*(x)w(x)\phi_B(x)dx.$$

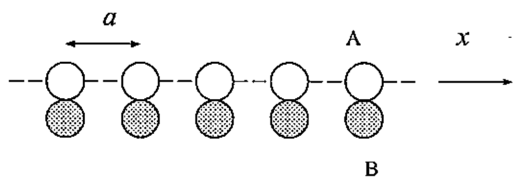


Figure 2: One-dimensional crystal composed of AB molecules.

[5] The Schrödinger equation of this one-dimensional molecular crystal is given by adding the potential of B ions, $W(x) = \sum_{j=1}^{M} w(x - ja)$, to $U(x)$ in eq. (1). Let us assume the solution in the form of

$$\Psi_k(x) = \sum_{j=1}^{M} e^{ik\cdot(ja)}\left(c_k\phi_A(x - ja) + d_k\phi_B(x - ja)\right). \tag{3}$$

Write down equations for c_k and d_k with the energy eigenvalue $E(k)$.

[6] Consider $E(k)$ introduced in question [5] according to the following procedure. First, find $E(k)$ for the case of $\delta = 0$, and plot $E(k)$ for $|E_A - E_B + 2\beta| \ll |\gamma|$ and $|E_A - E_B + 2\beta| \gg |\gamma|$, separately. Next, find $E(k)$ for $\delta \neq 0$. Assuming $|\delta| \ll |\gamma|$, plot $E(k)$ for $|E_A - E_B + 2\beta| \ll |\gamma|$ and $|E_A - E_B + 2\beta| \gg |\gamma|$, separately. Indicate E_A, E_B, β, γ, and δ in the plots.