

Department of Applied Physics

Entrance Examination Booklet

Specialized Subjects

(Choose 3 Problems among the 4 Problems in this Booklet)

September 1 (Tuesday) 13:00 – 16:00, 2009

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Choose three problems among the four problems in this booklet, and answer the three problems.
4. Use one answer sheet for each problem (three answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. Do not take this booklet and the answer sheets with you after the end of the examination.

Applicant Number	No.
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write down your applicant number above

Problem 1

A magnetic moment under a magnetic field precesses around the field direction. To show this in quantum mechanics, let us calculate the time evolution of the expectation value of spin operator $\mathbf{S}=(S_x, S_y, S_z)$ in the Heisenberg and Schrödinger pictures. Here, assume that the magnetic flux density given by $\mathbf{B}_0 = (0, 0, B_0)$ is time-independent and the magnetic moment $\boldsymbol{\mu}$ is given as $\boldsymbol{\mu} = \gamma\hbar\mathbf{S}$, where γ is the gyromagnetic ratio and \hbar is the Planck constant h divided by 2π .

- [1] Write down the Hamiltonian describing this system.
- [2] Answer the following questions, regarding \mathbf{S} as a time-dependent operator (Heisenberg picture).
 - [2.1] Write down the Heisenberg equation of motion for S_x .
 - [2.2] Give the expression for the time dependence of $\langle S_x \rangle$, which is the expectation value of S_x . Here, assume $\langle S_x \rangle = 0$ at the time $t=0$.
 - [2.3] Give the expressions for the time dependences of $\langle S_z \rangle$ and $\langle S_x^2 + S_y^2 \rangle$ in the similar manner, and prove that $\langle \mathbf{S} \rangle$ precesses, using the results for $\langle S_x \rangle$, $\langle S_z \rangle$ and $\langle S_x^2 + S_y^2 \rangle$.
- [3] Answer the following questions, regarding \mathbf{S} as a time-independent operator (Schrödinger picture).
 - [3.1] A spin state at a time t , $|t\rangle$, is given by operating the time evolution operator on an initial state $|0\rangle$. Calculate $\langle S_x \rangle$ for the state $|t\rangle$ and show that the resultant time dependence of $\langle S_x \rangle$ is the same as that obtained in [2.2]. Use the following equation, which holds for an arbitrary real number θ :
$$\exp(-i\theta S_z)S_x \exp(i\theta S_z) = S_x \cos \theta + S_y \sin \theta \quad (1)$$
 - [3.2] Prove Eq.(1) in the following procedure.
 - i) Assuming $f(\theta) = \exp(-i\theta S_z)S_x \exp(i\theta S_z)$, give the expressions for $df(\theta)/d\theta$ and $d^2f(\theta)/d\theta^2$. In particular, the latter is expressed by f alone.
 - ii) By solving the differential equations obtained above, prove Eq.(1).

Problem 2

Temperature (T) dependence of the constant-volume specific heat C_V of a crystal is often shown by plotting C_V/T as a function of T^2 at low temperatures. Consider the reason for this plotting according to the following procedures.

Among the contributions to C_V from electrons and lattice vibrations, let us first consider the contribution from electrons. For simplicity, we neglect the electron-electron interaction and the coupling between electrons and lattice vibrations. Let V be the crystal volume, ε_F the chemical potential of electrons at $T = 0$ (the Fermi energy), k_B the Boltzmann constant, and \hbar the Planck constant h divided by 2π .

- [1] Write down the occupation probability for a quantum state of electrons with energy ε , $f(\varepsilon)$, and illustrate its form at T . Here, let μ be the chemical potential of electrons at T .
- [2] Suppose that the number of states for electrons per volume, energy, and spin is a constant D_0 near $\varepsilon = \varepsilon_F$. Show that the chemical potential μ is independent of T at low T . You may use the low- T expansion valid for $f(\varepsilon)$ in [1] and an arbitrary function $g(\varepsilon)$;

$$\int_{-\infty}^{\infty} f(\varepsilon)g(\varepsilon)d\varepsilon = \int_{-\infty}^{\mu} g(\varepsilon)d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \left. \frac{dg}{d\varepsilon} \right|_{\varepsilon=\mu} + \frac{7\pi^4}{360}(k_B T)^4 \left. \frac{d^3g}{d\varepsilon^3} \right|_{\varepsilon=\mu} + \dots$$

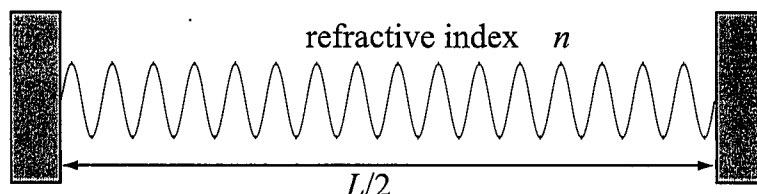
- [3] From T dependence of the internal energy of electrons at low T , calculate the contribution to C_V from electrons.

Next, consider the contribution from lattice vibrations. The quantum of the lattice vibrations is called phonon in the quantum theory, and the phonons obey the Bose-Einstein statistics with zero chemical potential. It is known that the contribution to C_V from phonons is dominated by acoustic modes at low T . Suppose that a unit cell contains a single atom.

- [4] Write down the occupation probability for a quantum state of phonons with angular frequency ω , $n(\omega)$, and illustrate its form at T .
- [5] Assuming that the acoustic mode has a linear dispersion relation $\omega = vk$ for simplicity (v is the sound velocity, $k = |\mathbf{k}|$, \mathbf{k} is the wave vector), calculate the number of states for phonons per angular frequency and mode.
- [6] Under the assumption in [5], from T dependence of the internal energy of phonons at low T , calculate the contribution to C_V from phonons. Here, suppose that there are three acoustic modes and all of the three have the same linear dispersion relation $\omega = vk$. You may use the relations $\int_0^{\infty} \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$ and $\int_0^{\infty} \frac{x^4 e^x}{(e^x-1)^2} dx = \frac{4\pi^4}{15}$.
- [7] Based on the results in [3] and [6], explain why the specific heat of a crystal is shown by plotting C_V/T as a function of T^2 , and describe what physical quantities can be read from the plot. In the case that C_V/T becomes zero in the limit of $T \rightarrow 0$, describe the property of the crystal.

Problem 3

There is an optical cavity consisting of two parallel mirrors, as shown in the figure. c , L , ω , and n denote the speed of light, the geometrical round-trip cavity length, the angular frequency of light, and the refractive index inside the cavity, respectively. There is no dispersion outside the cavity.



- [1] Show the phase change θ of the optical electric field after one round trip by using c , ω , L , and n . Neglect the phase slip by the reflection for simplicity.
- [2] ω_m is defined as the angular frequency of the longitudinal mode of the cavity which satisfies $\theta = 2\pi m$, where m is a natural number. Show ω_m by using c , L , n , and m .

The cavity outputs a part of the internal field through a mirror. The superposition of some longitudinal modes with a suitable phase relation makes an amplitude modulation of the electric field, which looks like a pulse train. Let us consider the electric field made by superposition of three neighboring longitudinal modes.

- [3] First, consider the superposition of three plane waves that have three neighboring longitudinal-mode frequencies in a non-dispersive medium with $n = 1$. Time dependences of the electric field component at a certain position outside the cavity are given by $E_{m-1}(t) = \frac{1}{2}E_0 \cos \omega_{m-1}t$, $E_m(t) = E_0 \cos \omega_m t$, and $E_{m+1}(t) = \frac{1}{2}E_0 \cos \omega_{m+1}t$. Show that the superposed field $E(t) = E_{m-1}(t) + E_m(t) + E_{m+1}(t)$ can be described by the product of an envelope function $A(t)$ and a carrier wave $\cos \omega_m t$, and illustrate the shape of $A(t)$. Derive the time interval between neighboring peaks of $A(t)$.
- [4] Suppose that n depends on ω and the group velocity $v_g = \frac{d\omega}{dk}$ is constant in the wavelength region considered here. Then the wave number can be written as $k = k_0 + \frac{\omega}{v_g}$ (k_0 is a constant). Express n in terms of c , k_0 , ω , and v_g .
- [5] Under this condition, show that ω_m can be written as $\omega_m = \omega_0 + m\omega_{rep}$, where m is a natural number.
- [6] Consider the superposed electric field of $E_{m-1}(t) = (1/2)E_0 \cos \omega_{m-1}t$, $E_m(t) = E_0 \cos \omega_m t$, and $E_{m+1}(t) = (1/2)E_0 \cos \omega_{m+1}t$ when the optical angular frequency can be written as $\omega_m = \omega_0 + m\omega_{rep}$. Calculate the time interval of successive peaks of the superposed field, and describe the physical meaning of this quantity.
- [7] The phase of the carrier wave at i -th peak of the envelope function is defined as ϕ_i . Express the phase difference between successive peaks $\Delta\phi = |\phi_{i+1} - \phi_i|$ in terms of ω_0 and ω_{rep} , ignoring integer multiple of 2π . Show that $\Delta\phi$ is caused by the difference between the group- and phase-velocity.

Problem 4

Answer the following questions on the X-ray diffraction (wavelength 1.5 \AA from a silicon crystal. Numerical values must have an accuracy of two significant digits. The following physical constants can be used if necessary.

Planck's constant $6.63 \times 10^{-34} \text{ J}\cdot\text{s}$

speed of light $3.0 \times 10^8 \text{ m/s}$

Boltzmann's constant $1.38 \times 10^{-23} \text{ J/K}$

elementary charge $1.60 \times 10^{-19} \text{ C}$

mass of neutron $1.67 \times 10^{-27} \text{ kg}$

- [1] Calculate the energy of the used X-ray in eV.
- [2] If we perform diffraction experiments using neutron beams with the same wavelength, how much is the kinetic energy of the neutron in Kelvin?
- [3] Let us consider a diffraction from a set of lattice planes, which are equidistant and parallel to each other, in a crystal lattice defined by mutually orthogonal primitive translation vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ ($|\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = a$). When one plane contains the origin and the neighboring plane intersects each axis as shown in Fig. 1, h, k, l are called Miller indices of the lattice plane and the corresponding plane is called (hkl) -plane. Calculate the distance between neighboring (hkl) -planes, $d(hkl)$. Orthogonality of the (hkl) -plane and the reciprocal lattice vector $\mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3$ may be used if necessary. Here, $\mathbf{b}_1, \mathbf{b}_2,$ and \mathbf{b}_3 are primitive translation vectors of the reciprocal lattice.

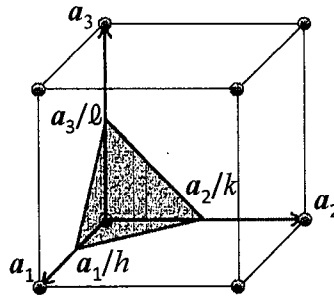


Figure 1: Crystal plane and Miller index

- [4] A silicon crystal has the diamond structure, which is constructed by placing two silicon atoms on each lattice point of a face-centered-cubic lattice as shown in Fig. 2. Calculate the relative diffraction intensity from the (hkl) -planes of the silicon crystal using h, k, l . The structure factor $F(\mathbf{K})$ of the crystal consisting of n silicon atoms in the unit cell can be expressed as follows:

$$F(\mathbf{K}) = \sum_{j=1}^n f e^{-i\mathbf{K}\cdot\mathbf{r}_j}. \quad (1)$$

Here, \mathbf{K} is the scattering vector, \mathbf{r}_j is the coordinate of j -th atom, and f is the atomic scattering factor of the silicon atom neglecting the scattering vector dependence.

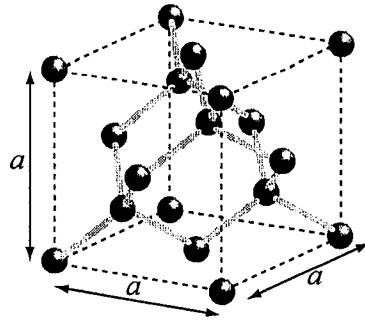


Figure 2: Diamond structure

- [5] What determines the atomic scattering factor f ?
- [6] With the lattice parameter of silicon $a = 5.43 \text{ \AA}$, calculate the second smallest diffraction angle (angle between incident and diffracted X-rays) with a non-zero diffraction intensity, and the corresponding Miller indices. If necessary, you can use Fig. 3.

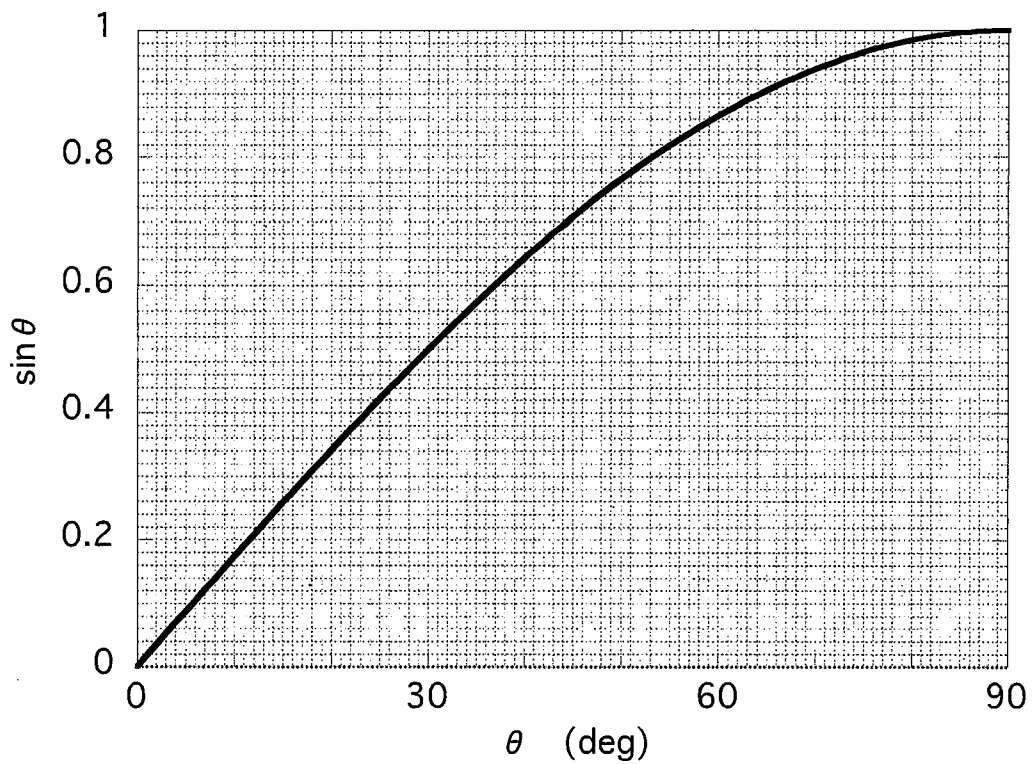


Figure 3: Sine curve

- [7] Show the Ewald construction for the situation described in [6]. Care must be taken to keep the directions of incident and diffracted X-rays and the reciprocal lattice semi-quantitatively correct.